The Interaction and Möbius Representations of Fuzzy Measures on Finite Spaces, k-Additive Measures: A Survey

Michel GRABISCH
Thomson-CSF, Corporate Research Laboratory
Domaine de Corbeville
91404 Orsay Cedex, France
email grabisch@thomson-lcr.fr

Abstract

This paper presents a synthesis of results on the concept of k-additive measures and the notion of representation of a fuzzy measure, since their introduction by the author in 1996. The collection of results will limit itself to results which are not connected with multicriteria decision making, since those are presented in a companion paper in this book, co-authored with Marc Roubens. The paper presents the concept of representation of a set function over a finite set, and introduces the well-known Möbius representation, and the interaction representation. Properties and links among these representations are given, and the notion of k-additive measure is introduced.

1 Introduction

Mathematically speaking, monotonic measures (or fuzzy measures, non-additive measures, capacities, games) can be considered from different points of view:

- Monotonic measures extend classical measures, by removing the additivity property. Underlying spaces are usually infinite, and some suitable algebra of sets is defined. Integration theory follows naturally. This is the *measure-theoretic viewpoint*, leading to non-additive measure theory, as developed in the books of Denneberg [6], Wang and Klir [34], and Pap [27].

- monotonic measures are particular cases of set functions, where usually monotonicity is not assumed. In the case of finite spaces, they relate naturally to combinatorics (see e.g. Rota [28]), to cooperative game theory [31], and also to pseudo-Boolean functions [19], which are used in complexity analysis.

Needless to say, the kinds of mathematics underlying these two points of view are completely different. In this paper, we adopt the second one, thus viewing fuzzy measures as particular set functions, and applying tools of set functions to them. In this framework, the Möbius transform, the concept of interaction representation, and even the concept of k-additive measure arise naturally. Further refinements are guided by more applicative concerns, like decision theory. We will present some of them in this paper, although we refer the reader to the companion paper in this book, co-authored with Marc Roubens, for any issue related to multicriteria decision making.

This survey paper collects results scattered in many papers, mainly [7, 8, 11, 10, 12, 14, 13, 24, 16, 29]. The reader is referred to them for further details and proofs.

Throughout the paper, we assume a finite space N with n elements, denoted simply $1, 2, \ldots, n$ if there is no fear of ambiguity. In a similar way, $s, t, \ldots$ will denote the cardinality of subsets $S, T, \ldots$ of N. We denote by $\wedge, \vee$ the minimum and maximum operators on the real line.

2 Representations of a set function

We consider real valued set functions $v : \mathcal{P}(N) \longrightarrow$, and several particular cases. Set functions vanishing on the empty set are called *games*, while *fuzzy measures*, which we will denote by μ, refer to games which are monotonic with respect to inclusion, i.e.

$$A \subset B \Rightarrow \mu(A) \leq \mu(B).$$

Consequently, fuzzy measures assume only positive values. In applications, it is often required in addition that $\mu(N) = 1$.

For any set function v, the *dual set function* or *conjugate set function* of v is defined by

$$\bar{v}(S) := v(N) - v(S^c), \quad \forall S \subset N, \tag{1}$$

where S^c is the complement set of S.

We introduce some special properties of fuzzy measures. A fuzzy measure is said to be

- *k-monotone* ($k \geq 2$) [3] if for all families of k subsets $A_1, \ldots, A_k$ in N,

$$\mu(\bigcup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subset \{1,\ldots,k\}} (-1)^{|I|+1} \mu(\bigcap_{i \in I} A_i). \tag{2}$$

Also, 1-monotonicity is defined as monotonicity.

- *totally monotone* if it is k-monotone for any $k \geq 2$.
- *k-alternating* ($k \geq 2$) if for all family of k subsets $A_1, \dots, A_k$ in N,

$$\mu(\bigcap_{i=1}^{k} A_i) \leq \sum_{\emptyset \neq I \subset \{1,\dots,k\}} (-1)^{|I|+1} \mu(\bigcup_{i \in I} A_i). \tag{3}$$

2-monotone measures are also called *supermodular* or *convex*, while 2-alternating measures are called *submodular*. If μ is k-monotone, then $\bar{\mu}$ is k-alternating, and reciprocally.

Any set function v is defined unambiguously by the collection of 2^n real numbers $v(\emptyset), v(\{1\}), v(\{2\}), \dots, v(N)$. One can imagine any transformation $\mathcal{T}$ (e.g. a linear one) of these coefficients to get another set of coefficients. If the transformation is invertible, then it becomes equivalent to give $\mathcal{T}(\{v(S)\}_{S \subset N})$ instead of $\{v(S)\}_{S \subset N}$. In this case, we say that $\mathcal{T}(v)$ is a *representation* of v. We give some examples of representation below.

Möbius representation: in combinatorics, the *Möbius transform* is well-known (see e.g. Rota [28]), and has been rediscovered many times. For any set function v, its Möbius transform m^v is defined by:

$$m^v(S) := \sum_{T \subset S} (-1)^{s-t} v(T), \quad \forall S \subset N. \tag{4}$$

The inverse transform is the *Zeta transform*, expressed by:

$$v(S) = \sum_{T \subset S} m^v(T), \quad \forall S \subset N. \tag{5}$$

m^v is called the *Möbius representation* of v. In game theory, this corresponds to the *dividends* of a game. As we will see in the sequel, these are the coefficients of the decomposition of a game w.r.t. the unanimity games. In Dempster-Shafer theory of evidence, this corresponds also to the *basic probability mass assignment* [30].

interaction representation: it has been proposed by Grabisch [8], and is defined for any set function v by:

$$I^v(S) := \sum_{T \subset N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{K \subset S} (-1)^{s-k} v(K \cup T), \forall S \subset N. \tag{6}$$

This complicated definition extends in fact the Shapley value ϕ^v [31] and the interaction index I_{ij} for a pair of elements i, j in N, introduced by Murofushi and Soneda [25]. They are defined by

$$\phi^v(i) := \sum_{S \subset N \setminus i} \frac{(n-s-1)!s!}{n!} [v(S \cup \{i\}) - v(S)], \forall i \in N \tag{7}$$

$$I_{ij} := \sum_{S\subset N\setminus\{i,j\}} \frac{(n-s-2)!s!}{(n-1)!}[v(S\cup\{i,j\})-v(S\cup\{i\})-v(S\cup\{j\})+v(S)], \tag{8}$$

It is easy to see that $I^v(\{i\}) = \phi^v(i)$ and $I^v(\{i,j\}) = I_{ij}$.

Since formula (6) is invertible (see below), I^v is indeed a representation of v, and it is often referred as the *interaction index*.

The interaction index has a natural interpretation in the framework of cooperative game theory and multicriteria decision making (see the companion paper in this book for more detail on this). It has been axiomatized by Grabisch and Roubens [18].

Banzhaf interaction representation: proposed by Roubens [29] in order to extend the Banzhaf value [1], in a way similar as above. It is defined by:

$$J(S) := \frac{1}{2^{n-s}} \sum_{T\subset N\setminus S} \sum_{K\subset S} (-1)^{s-k} v(K\cup T), \forall S \subset N. \tag{9}$$

co-Möbius representation: for any set function v, it is defined by

$$\check{m}^v(S) := \sum_{T\supset N\setminus S} (-1)^{n-t} v(T) = \sum_{T\subset S} (-1)^t v(N\setminus T), \forall S \subset N. \tag{10}$$

It will be seen below that $\check{m}^v$ corresponds to the *commonality function* of Shafer [30]. The analogy with the Möbius transform can be noticed.

In the sequel, we will show how tight are the links between all these representations.

3 Pseudo-Boolean functions and their extensions

This section relies essentially on [16].

3.1 Definition

Any function $f : \{0,1\}^n \longrightarrow$ is a said to be a *pseudo-Boolean function*. By making the usual bijection between $\{0,1\}^n$ and $\mathcal{P}(N)$, it is clear that pseudo-Boolean functions on $\{0,1\}^n$ coincide with real-valued set functions on N. More specifically, if we define for any subset $S \subset N$ the vector $1_S = [(1_S)_1 \cdots (1_S)_n]$ in $\{0,1\}^n$ by $(1_S)_i = 1$ if $i \in S$, and 0 otherwise, then for any set function v we can define its associated pseudo-Boolean function f by

$$f(1_S) := v(S), \quad \forall S \subset N,$$

and reciprocally. It has been shown by Hammer and Rudeanu [20] that any pseudo-Boolean function can be written in a multilinear form:

$$f(x) = \sum_{T \subset N} a(T) \prod_{i \in T} x_i, \quad \forall x \in \{0,1\}^n. \tag{11}$$

It is easy to see that the coefficients $a(T)$ coincide with the Möbius transform of the corresponding v (compare (5) and (11)). The monomials $\prod_{i\in T} x_i$ are particular pseudo-Boolean functions, whose corresponding set functions are called *unanimity games* in game theory, denoted by u_T. They are characterized by the property $u_T(S) = 1$ iff $S \supset T$, and $u_T(S) = 0$ otherwise. In terms of game theory, equation (11) gives the decomposition of a game on the basis of unanimity games.

Note that (11) can be put in an equivalent form, which is

$$f(x) = \sum_{T \subset N} a(T) \bigwedge_{i \in T} x_i, \quad \forall x \in \{0,1\}^n. \tag{12}$$

3.2 Derivative of a pseudo-Boolean function

It is useful to introduce the concept of *derivative* of a pseudo-Boolean function f. The *(first) derivative* of f w.r.t. i at point $x = [x_1 \cdots x_n]$ is defined by

$$\Delta_i f(x) := f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n). \tag{13}$$

Note that $\Delta_i f$ depends no more on x_i. More generally, the *(sth) derivative* w.r.t $S \subset N$, $S \neq \emptyset$ (or S-derivative) at point x is defined recursively by:

$$\Delta_S f(x) := \Delta_i(\Delta_{S\setminus i} f(x)) \tag{14}$$

for any $i \in S$, and $\Delta_\emptyset f = f$. This definition is unambiguous, and $\Delta_S f$ depends no more on the variables contained in S. It is easy to show that

$$\Delta_S f(1_T) = \sum_{L \subset S} (-1)^{s-l} v(L \cup T), \forall S \subset N, \forall T \subset N \setminus S. \tag{15}$$

Hence we obtain immediately that the interaction indices have a simple expression in terms of S-derivatives:

$$\begin{aligned} I(S) &= \frac{1}{n-s+1} \sum_{T \subset N \setminus S} \binom{n-s}{t}^{-1} \Delta_S f(1_T) \\ J(S) &= \frac{1}{2^{n-s}} \sum_{T \subset N \setminus S} \Delta_S f(1_T). \end{aligned}$$

We can also easily show that

$$\begin{aligned} m^v(S) &= \Delta_S f(0_N) \\ \check{m}^v(S) &= \Delta_S f(1_N). \end{aligned}$$

where $0_N := [0 \cdots 0]$, and $1_N = [1 \cdots 1]$.

3.3 Extension of a pseudo-Boolean function

An *extension* of a pseudo-Boolean function f is any function g defined on the whole hypercube $[0,1]^n$ such that $g(x) = f(x)$ on the vertices. An obvious way to do this is to extend expressions (11) and (12) to the whole hypercube. The first one is called the *multilinear extension*, while the second is the *Lovász extension*.

The multilinear extension of f, given by

$$\bar{f}(x) := \sum_{T \subset N} a(T) \prod_{i \in T} x_i, \quad \forall x \in [0,1]^n, \tag{16}$$

is the only multilinear function which extends f, hence its name (see Owen [26]). It performs the classical linear interpolation of f in $[0,1]^n$.

It can be verified that the S-derivative of $\bar{f}$ in the classical sense is the multilinear extension of $\Delta_S f$. The following result can be shown.

Proposition 1 *Let v be a set function and $\bar{f}$ its multilinear extension. Then, for all $S \subset N$,*

(i) $J(S) = \Delta_S \bar{f}\left(\left(\frac{1}{2}\right)_N\right)$

(ii) $I(S) = \int_0^1 \Delta_S \bar{f}(x_N) dx$

(iii) $J(S) = \int_{[0,1]^{n-s}} \Delta_S \bar{f}(x) dx$

where $x_N := [xx \cdots x]$ for any real number x.

Comparing formula *(iii)* with (9), we see that the Banzhaf interaction index is the average of the S-derivative of the pseudo-Boolean function, and of its multilinear extension too. The Shapley interaction index is by contrast the average along the main diagonal only.

The Lovász extension of f [21, 32] is given by

$$\hat{f}(x) := \sum_{T \subset N} a(T) \bigwedge_{i \in T} x_i, \quad \forall x \in [0,1]^n. \tag{17}$$

Defining the simplex $\mathcal{B}_\pi = \{x \in [0,1]^n | x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}$, the Lovász extension is the unique affine function which interpolates f between the

$n+1$ vertices of $\mathcal{B}_\pi$. By contrast, the multilinear extension interpolates between *all* vertices.

As above, its derivative coincides with the Lovász extension of $\Delta_S f$.

The following can be shown.

Proposition 2 *Let v be a set function and $\hat{f}$ its Lovász extension. Then, for all $S \subset N$,*

$$I(S) = \int_{[0,1]^n} \Delta_S \hat{f}(x) dx. \quad (18)$$

4 Conversion formulas between various representations

We give in this section all conversion formulas between v, $\check{m}$, m, I and J (superscript v is omitted). We begin with the most usual ones.

We consider first the Möbius and Shapley interaction representations. Table 1 gives all formulas for conversion between v, m and I. Conversion

	v	m	I
$v(S) =$	$v(S)$	$\sum_{T \subset S} m(T)$	$\sum_{T \subset N} \beta^{\lvert T\rvert}_{\lvert S\cap T\rvert} I(T)$
$m(S) =$	$\sum_{T \subset S} (-1)^{s-t} v(T)$	$m(S)$	$\sum_{T \supset S} B_{t-s} I(T)$
$I(S) =$	$\sum_{T \subset N} \frac{(-1)^{s-t}}{(n-s+1)\binom{n-s}{t-s}} v(T)$	$\sum_{T \supset S} \frac{1}{t-s+1} m(T)$	$I(S)$

Table 1: Table of conversion between v, m and I

formulas between m and v have already been given at the beginning of the paper. Remark that contrary to (6), the expression of I w.r.t. v uses only one summation: in fact every $v(T)$ is used exactly once.

It is remarkable that the expression of I w.r.t m is so simple, compared to (6). In fact, originally, I was defined first in terms of m, as a remaining term in an approximation problem. Then its expression in terms of v was found, which happened to be a generalization of $\phi(i)$ and I_{ij} (see [8]). The inverse relation (of m in terms of I) interestingly leads to the introduction of the Bernoulli numbers B_k, defined by:

$$B_k := -\sum_{l=0}^{k-1} \frac{B_l}{k-l+1} \binom{k}{l}, k > 0, \quad (19)$$

and $B_0 = 1$. First numbers of the sequence are $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, etc. Despite the erratic behaviour of this sequence, it has remarkable properties. In particular, $B_{2k+1} = 0$ for all $k > 0$. The numbers β_k^l are defined from the Bernoulli numbers by

$$\beta_k^l := \sum_{j=0}^{k} \binom{k}{j} B_{l-j}, \quad k, l = 0, 1, 2, \dots \tag{20}$$

First values of β_k^l are

$k \setminus l$	0	1	2	3	4
0	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
1		$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{30}$
2			$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{15}$
3				0	$-\frac{1}{30}$
4					$-\frac{1}{30}$

This table has a property similar to the Pascal triangle, i.e. one coefficient β_{k+1}^{l+1} is the sum of the two above β_k^l, β_k^{l+1}. For other properties, see [10].

Except the conversion from I to μ, these relations have been proven in the original paper [11] in a cumbersome way. More elegant proofs have been provided by Denneberg in [7], and by Marichal in [16], the latter using multilinear extensions of pseudo-Boolean functions.

We turn to the conversion between v, m and the Banzhaf interaction, which are more straightforward to obtain, especially using a matrix formalism (see Roubens [29]). Formulas are shown in table 2.

	v	m
$J(S) =$	$\frac{1}{2^{n-s}} \sum_{T \subset N} (-1)^{s-t} v(T)$	$\sum_{T \supset S} \frac{1}{2^{t-s}} m(T)$

	J
$v(S) =$	$\sum_{T \subset N} \frac{1}{2^t} (-1)^{t-s} J(T)$
$m(S) =$	$\sum_{T \supset S} \left(-\frac{1}{2}\right)^{t-s} J(T)$

Table 2: Tables of conversion between v, m and J

We introduce now the co-Möbius representation $\check{m}$. Table 3 shows the relations with other representations. For the expression of $\check{m}$ in terms of I, the following is equivalent [10]:

$$\check{m} = \sum_{T \supset S} \beta_{t-s}^{t-s} I(T).$$

It is interesting to see that $\check{m}(S)$ is the sum of the Möbius coefficients for all supersets of S. Compared to $v(S)$, which is the sum over all subsets of S,

	v	m	I	J
$\check{m}(S) =$	$\sum_{T\supset N\setminus S}(-1)^{n-t}v(T)$	$\sum_{T\supset S} m(T)$	$\sum_{T\supset S}(-1)^{t-s}B_{t-s}I(T)$	$\sum_{T\supset S}\frac{1}{2^{t-s}}J(T)$

	$\check{m}$
$v(S) =$	$\sum_{T\subset N\setminus S}(-1)^{t}\check{m}(T)$
$m(S) =$	$\sum_{T\supset S}(-1)^{t-s}\check{m}(T)$
$I(S) =$	$\sum_{T\supset S}\frac{(-1)^{t-s}}{t-s+1}\check{m}(T)$
$J(S) =$	$\sum_{T\supset S}\left(-\frac{1}{2}\right)^{t-s}\check{m}(T)$

Table 3: Tables of conversion between $\check{m}$ and other representations

we see that $\check{m}$ is a kind of "complement" of v. In the case of fuzzy measures, this means that $\check{m}$ is anti-monotonic, and vanishes on the whole set, hence the term *co-measure* which has been employed in [10]. On the other hand, in evidence theory, $\check{m}$ is known as the *commonality function*, which has many interesting properties. Compare also the expressions for I and J in terms of m and $\check{m}$.

Lastly, we give the conversion formulæ between I and J in table 4. They can be obtained through the multilinear extension [16].

	I	J
$I(S) =$	$I(S)$	$\sum_{T\supset S}\frac{1+(-1)^{t-s}}{(t-s+1)2^{t-s+1}}J(T)$
$J(S) =$	$\sum_{T\supset S}\left(\frac{1}{2^{t-s-1}}-1\right)B_{t-s}I(T)$	$J(S)$

Table 4: Table of conversion between I and J

5 Transformations and matrices

As we explained, the formulas of section 4 can be obtained directly (but tediously!) from the definitions and elementary calculus over subsets, or in a more sophisticated way, using extensions of pseudo-Boolean functions. There are still two other ways (in fact essentially equivalent), which enlighten the deep structure underlying these representations. We present them briefly (see [7, 16] for more details).

The first way is to adopt the formalism of transformations and operators, used in combinatorics (see Berge [2]). An *operator* is a two-place set function $\Phi : \mathcal{P}(N) \times \mathcal{P}(N) \longrightarrow$. The multiplication $\star$ between operators and set functions is defined as follows, for every $A, B, C \subset N$

$$
\begin{aligned}
(\Phi \star \Psi)(A,B) &:= \sum_{C \subset N} \Phi(A,C)\Psi(C,B)\,, \\
(\Phi \star v)(A) &:= \sum_{C \subset N} \Phi(A,C)v(C)\,, \\
(v \star \Psi)(B) &:= \sum_{C \subset N} v(C)\Psi(C,B)\,.
\end{aligned}
$$

The Kronecker's delta

$$
\Delta(A,B) := \begin{cases} 1 & \text{if} \quad A = B \\ 0 & \text{else} \end{cases}
$$

is the unique neutral element from the left and from the right. If it exists, the inverse of Φ is denoted Φ^{-1}, satisfying $\Phi \star \Phi^{-1} = \Delta$, $\Phi^{-1} \star \Phi = \Delta$.

It can be shown that the family

$$
\mathcal{G} := \{\Phi : \mathcal{P}(N) \times \mathcal{P}(N) \to \ \mid \Phi(A,A) = 1 \, \forall A \subset N,\ \Phi(A,B) = 0 \text{ if } A \not\subset B\}
$$

of functions of two variables together with the operation $\star$ forms a group, and the inverse $\Phi^{-1} \in \mathcal{G}$ of $\Phi \in \mathcal{G}$ computes recursively through

$$
\begin{aligned}
\Phi^{-1}(A,A) &= 1, \\
\Phi^{-1}(A,B) &= -\sum_{A \subset C \subsetneq B} \Phi^{-1}(A,C)\Phi(C,B) \quad \text{if } A \subsetneq B.
\end{aligned}
$$

The *Zeta operator* $Z(A,B)$, defined by

$$
Z(A,B) := \begin{cases} 1 & \text{if} \quad A \subset B \\ 0 & \text{else} \end{cases}
$$

and its inverse the *Möbius operator* correspond to our previous definitions, i.e. with former notations,

$$
m = v \star Z^{-1}, \quad v = m \star Z. \tag{21}
$$

The next fundamental operator to introduce is the so-called *inverse Bernoulli operator* Γ:

$$
\Gamma(A,B) := \begin{cases} \frac{1}{|B \setminus A| + 1} & \text{if } A \subset B \\ 0 & \text{else} \end{cases} .
$$

The interaction representation is recovered by

$$
I = \Gamma \star m. \tag{22}
$$

We turn now to a special class of operators, satisfying

$$\Phi(A,B) = \Phi(\emptyset, B \setminus A) \quad \text{for } A \subset B, \tag{23}$$

i.e. they can be represented by an ordinary set function $\varphi(A) := \Phi(\emptyset, A)$, denoted with the corresponding small greek letter. In fact, the set of such operations forms an Abelian group, as well as the corresponding set of set functions:

$$\mathsf{g} := \{\varphi : \mathcal{P}(N) \to \ \mid \varphi(\emptyset) = 1\}$$

with operation $\star$ defined by

$$\varphi \star \psi(A) := \sum_{C \subset A} \varphi(C)\psi(A \setminus C), \quad A \subset N.$$

The neutral element δ of g is

$$\delta(A) := \begin{cases} 1 & \text{if} \quad A = \emptyset \\ 0 & \text{else} \end{cases},$$

and the inverse of φ is denoted $\varphi^{\star -1}$. Since Z and Γ have property (23), we can introduce the corresponding Zeta function and Bernoulli function:

$$\zeta(A) = 1 \quad \text{for all } A \in \mathcal{P},$$
$$\gamma(A) = \frac{1}{|A|+1}, \quad A \in \mathcal{P}.$$

If moreover φ is a function only of the cardinal of sets, then we call it a *cardinality function*, and the corresponding Φ a *cardinality operator*. Note that Z and Γ have also this property.

There is a general formula for the inverse of cardinality operators (which is also cardinal). More specifically, if $\varphi(A) = f(|A|)$, then $\varphi^{\star -1}(A) = f^{\star -1}(|A|)$, with $f^{\star -1}$ defined recursively by

$$\begin{aligned} f^{\star -1}(0) &:= 1\,, \\ f^{\star -1}(m) &:= -\sum_{k=0}^{m-1} \binom{m}{k} f(m-k) f^{\star -1}(k)\,, \quad m \in . \end{aligned} \tag{24}$$

With this formula, we get $\zeta^{\star -1}$ the Möbius function, and $\gamma^{\star -1}$ the Bernoulli function, giving rise to the Bernoulli numbers.

Coming back to the interaction representation, we have

$$I = \Gamma \star m = \Gamma \star (v \star Z^{-1}),$$

which can be rearranged to obtain $I = v \star \mathcal{I}$. $\mathcal{I}$ is called the *interaction operator*, and is not in the group $\mathcal{G}$.

The second way is to use matrices instead of operators. Considering the set of all subsets of N, one can consider a set function on N as a vector in $\mathcal{P}(N)$ (i.e. with 2^n lines), and operators $\Phi(A,B)$ as squared matrices of 2^n rows and columns. Thus, the multiplication $\star$ is now the vector/matrix or the matrix/matrix product. If a suitable order is chosen for the subsets of N, then the matrices corresponding to the above cited operators have a very interesting structure. Let us consider the following order:

$$O : \emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \ldots, N.$$

Then most of the operators above have a corresponding matrix which is called *fractal*, because the whole matrix can be reconstructed from an elementary cell, as follows

$$\begin{aligned} F_{(1)} &:= \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}, \quad f_i \in \,, \; i = 1,2,3,4 \\ F_{(k)} &:= \begin{bmatrix} f_1 F_{(k-1)} & f_2 F_{(k-1)} \\ f_3 F_{(k-1)} & f_4 F_{(k-1)} \end{bmatrix}. \end{aligned}$$

This is the case for all transformations between v, m, $\check{m}$ and J.

A second type of matrix is the so-called *upper-cardinality transformation*, which corresponds to the cardinality operators above. The general form of these matrix is given below.

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \emptyset & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \ldots \end{array} \\
\begin{array}{c} \emptyset \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \vdots \end{array} &
\begin{bmatrix}
c_0 & c_1 & c_1 & c_2 & c_1 & c_2 & c_2 & c_3 & \\
 & c_0 & & c_1 & & c_1 & & c_2 & \\
 & & c_0 & c_1 & & & c_1 & c_2 & \\
 & & & c_0 & & & & c_1 & \\
 & & & & c_0 & c_1 & c_1 & c_2 & \\
 & & & & & c_0 & & c_1 & \\
 & & & & & & c_0 & c_1 & \\
 & & & & & & & c_0 & \\
 & & & & & & & &
\end{bmatrix}
\end{array}$$

and in fact is completely defined by the sequence $c_0, c_1, \ldots, c_n$. This sequence corresponds to the function f above. If c_k is such that $c_k = \rho^k$ for any k, then it is also a fractal matrix. All transformations between m, $\check{m}$, I and J are upper-cardinal, of which sequences are given in table 5. A third type of matrix is the lower-cardinality matrix, which is just the transpose of an upper-cardinality matrix.

This formalism gives an elegant and quick way of computing the transformations.

	m	$\check{m}$	I	J
m	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = (-1)^k$	$c_k = B_k$	$c_k = \left(-\frac{1}{2}\right)^k$
$\check{m}$	$c_k = 1$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = (-1)^k B_k$	$c_k = \left(\frac{1}{2}\right)^k$
I	$c_k = \dfrac{1}{k+1}$	$c_k = \dfrac{(-1)^k}{k+1}$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = \dfrac{1+(-1)^k}{(k+1)2^{k+1}}$
J	$c_k = \left(\frac{1}{2}\right)^k$	$c_k = \left(-\frac{1}{2}\right)^k$	$c_k = (\frac{1}{2^{k-1}} - 1)B_k$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$

Table 5: Cardinality sequences for equivalent representations between m, $\check{m}$, I, and J

6 Main properties of the representations

We give here essential properties of the different representations. Proofs and other details can be found in [7, 11, 24].

We begin by properties involving the conjugate set function.

Property 1 *For any set function v and its conjugate $\bar{v}$, we have*

$$\check{m}^{\bar{v}}(S) = (-1)^{s+1} m^v(S), \quad \forall S \subset N, S \neq \emptyset \tag{25}$$

$$m^{\bar{v}}(S) = (-1)^{s+1} \sum_{T \supset S} m^v(T), \quad \forall S \subset N, S \neq \emptyset. \tag{26}$$

Equation (25) is obtained directly from (10) as follows:

$$\begin{aligned} \check{m}^{\bar{v}}(S) &= \sum_{T \subset S} (-1)^t \bar{v}(T^c) \\ &= \sum_{T \subset S} (-1)^t (v(N) - v(T)) \\ &= -\sum_{T \subset S} (-1)^t v(T) = (-1)^{s+1} m^v(S). \end{aligned}$$

Equation (26) has been proven in [11] with a lengthy proof. As indicated by Roubens, the result is immediate from (25) since it implies

$$m^{\bar{v}}(S) = (-1)^{s+1} \check{m}^v(S) = (-1)^{s+1} \sum_{T \supset S} m^v(T).$$

Property 2 *For any set function v and its conjugate $\bar{v}$, we have*

$$\begin{aligned} I^{\bar{v}}(\emptyset) &= v(N) - I^v(\emptyset) \\ I^{\bar{v}}(S) &= -(-1)^s I^v(S), \quad \forall S \subset N, S \neq \emptyset. \end{aligned}$$

The two following properties concern the bounds of the interaction and Möbius representations.

Property 3 *Let μ be a fuzzy measure with $\mu(N) = 1$. Then the interaction index I_{ij} ranges in $[-1, +1]$. $I_{ij} = 1$ if and only if $\mu = u_{\{i,j\}}$, the unanimity game for the pair i, j. Similarly, $I_{ij} = -1$ if and only if $\mu = \bar{u}_{\{i,j\}}$, the dual measure of the unanimity game for the pair i, j.*

Property 4 *Let $S \subset N$. Then the upper bound $U(S)$ of $m(S)$, $I(S)$, and $J(S)$ taken over all fuzzy measures with $\mu(N) = 1$ is the same for all three representations, and is given by*

$$U(S) = \sum_{j=0}^{l_s} \binom{s}{j} (-1)^j = (-1)^{l_s} \binom{s-1}{l_s}, \tag{27}$$

with l_s defined as follows

(i) $l_s = \frac{s}{2}$ *if* $s \equiv 0 (\mathrm{mod} \quad 4)$

(ii) $l_s = \frac{s-1}{2}$ *if* $s \equiv 1 (\mathrm{mod} \quad 4)$

(iii) $l_s = \frac{s}{2} - 1$ *if* $s \equiv 2 (\mathrm{mod} \quad 4)$

(iv) $l_s = \frac{s-3}{2}$ *or* $l_s = \frac{s+1}{2}$ *if* $s \equiv 3 (\mathrm{mod} \quad 4)$.

Moreover, the bounds are reached by the fuzzy measure $\hat{\mu}$ defined by, for any $T \neq \emptyset$:

$$\hat{\mu}_s(T) = \begin{cases} 1, & \text{if } |T| \geq l_s \\ 0, & \text{if } |T| < l_s \end{cases}$$

Similarly, the lower bound $L(S)$ is given by the same expression, but with a different definition of l_s:

(i) $l_s = \frac{s}{2} - 1$ *if* $s \equiv 0 (\mathrm{mod} \quad 4)$

(ii) $l_s = \frac{s-3}{2}$ *or* $l_s = \frac{s+1}{2}$ *if* $s \equiv 1 (\mathrm{mod} \quad 4)$

(iii) $l_s = \frac{s}{2}$ *if* $s \equiv 2 (\mathrm{mod} \quad 4)$

(iv) $l_s = \frac{s-1}{2}$ *if* $s \equiv 3 (\mathrm{mod} \quad 4)$.

Note that the bounds are the same for m, I, and J.

The three following properties concern monotonicity and bounds of the set function. They ensure that a given set function, which is known only through either m, I, or J, is a fuzzy measure, i.e. monotonic with respect to inclusion, vanishing on the empty set, and being 1 on the whole set. The first result is due to Chateauneuf and Jaffray [3], the second one to Grabisch [11], and the third one to Roubens [17].

Property 5 *A set of 2^n coefficients $m(S)$, $S \subset N$ corresponds to the Möbius representation of a fuzzy measure if and only if*

(i) $m(\emptyset) = 0$, $\sum_{S\subset N} m(S) = 1$,

(ii) $\sum_{i\in T\subset S} m(T) \geq 0$, *for all* $S \subset N$, *for all* $i \in S$.

Property 6 *A set of 2^n coefficients $I(S)$, $S \subset N$ corresponds to the (Shapley) interaction representation of a fuzzy measure if and only if*

(i) $\displaystyle\sum_{S\subset N} B_s I(S) = 0$,

(ii) $\displaystyle\sum_{i\in N} I(\{i\}) = 1$,

(iii) $\displaystyle\sum_{S\subset N\setminus i} \beta^{|S|}_{|S\cap T|} I(S\cup\{i\}) \geq 0, \forall i \in N, \forall T \subset N \setminus \{i\}$.

Property 7 *A set of 2^n coefficients $J(S)$, $S \subset N$ corresponds to the (Banzhaf) interaction representation of a fuzzy measure if and only if*

(i) $\displaystyle\sum_{S\subset N} \left(-\frac{1}{2}\right)^s J(S) = 0$,

(ii) $\displaystyle\sum_{S\subset N} \left(\frac{1}{2}\right)^s J(S) = 1$,

(iii) $\displaystyle\sum_{S\subset N\setminus i} \left(\frac{1}{2}\right)^s (-1)^{s-t} J(S\cup\{i\}) \geq 0, \forall i \in N, \forall T \subset N \setminus \{i\}$.

Lastly, we give some properties linked with k-monotonicity.

Property 8 *If μ is a k-monotone fuzzy measure for a given $k \geq 1$ (resp. k-alternating, $k \neq 1$), then for each $S \subset N$ with $s \leq k$, $I(S) \geq 0$ (resp. ≤ 0 for s even, and ≥ 0 for s odd).*

Property 9 *A fuzzy measure is totally monotone if and only if $m(S) \geq 0$ for any $S \subset N$. If a fuzzy measure is k-monotone, then $m(S) \geq 0$ for any S such that $2 \leq s \leq k$.*

This result is due to Chateauneuf and Jaffray [3], while the first assertion is well known (see e.g. Shafer [30]).

Property 10 *Let μ be a fuzzy measure with $\mu(N) = 1$. If μ is k-monotone for a given $k \geq 1$, then*

$$I(S) \leq 1, \quad \forall S \subset N, s \leq k+1.$$

Similarly, if μ is k-alternating, $k \neq 1$, then for all $S \subset N$, $s \leq k+1$,

$$I(S) \leq 1, \text{ if } s \text{ odd}, \quad I(S) \geq -1, \text{ if } s \text{ even}.$$

7 The Choquet integral

So far, we have been concerned mainly with set functions and sometimes with fuzzy measures. Going back to the measure-theoretic point of view for a while, we could deal with integration with respect to a fuzzy measure. The Choquet integral [4] extends the classical Lebesgue integral for non-classical measures, such as fuzzy measures. As we deal with the finite case, we give its definition only in this framework, avoiding a thorough development, which can be found in this book in the papers of Denneberg, and Murofushi and Sugeno. We will see however that we can in fact remain in the framework of pseudo-Boolean functions, without referring to fuzzy measure theory.

Definition 1 *Let f be a positive real-valued function on N, and v a set function. Let us denote $f(i)$ by f_i, for every i in N. Then the Choquet integral of f with respect to v is given by*

$$\mathcal{C}_v(f) := \sum_{i=1}^{n} [f_{(i)} - f_{(i-1)}] v(A_{(i)}), \tag{28}$$

where $\cdot_{(i)}$ indicates a permutation on N so that $f_{(1)} \leq f_{(2)} \leq \cdots \leq f_{(n)}$, and $A_{(i)} := \{(i), \ldots, (n)\}$. Also $f_{(0)} := 0$.

For real-valued functions, the definition is extended as follows:

$$\mathcal{C}_v(f) := \mathcal{C}_v(f^+) - \mathcal{C}_{\bar{v}}(f^-), \tag{29}$$

where f^+, f^- are the positive and negative parts of f, such that $f = f^+ - f^-$.

It is interesting to express the Choquet integral under the various representations of v. The result concerning m has been shown to hold by Chateauneuf and Jaffray [3] (also by Walley [33]), extending Dempster [5], the result concerning J and $\check{m}$ has been proven by Roubens [17], while Grabisch [9] proved the one concerning I.

Property 11 *Let v be a set function, $m, \check{m}, I, J$ their Möbius, co-Möbius and interaction representations, and f a positive real-valued function on N. Then the Choquet integral of f w.r.t. v is expressed by:*

$$\mathcal{C}_v(f) = \sum_{S\subset N} m(S) \bigwedge_{i\in S} f_i \tag{30}$$

$$\mathcal{C}_v(f) = \sum_{S\subset N, S\neq\emptyset} (-1)^{s+1}\check{m}(S) \bigvee_{i\in S} f_i \tag{31}$$

$$\mathcal{C}_v(f) = \sum_{S\subset N} \left[\sum_{T\supset S} B_{t-s} I^+(T)\right] \bigwedge_{i\in S} f_i + \sum_{S\subset N, S\neq\emptyset} (-1)^{s+1} \left[\sum_{T\supset S} \beta^{t-s}_{t-s} I^-(T)\right] \bigvee_{i\in S} f_i \tag{32}$$

$$\mathcal{C}_v(f) = \sum_{S\subset N} \left[\sum_{T\supset S} \left(-\frac{1}{2}\right)^{t-s} J^+(T)\right] \bigwedge_{i\in S} f_i + \sum_{S\subset N, S\neq\emptyset} (-1)^{s+1} \left[\sum_{T\supset S} \left(\frac{1}{2}\right)^{t-s} J^-(T)\right] \bigvee_{i\in S} f_i \tag{33}$$

$$\tag{34}$$

with I^+ indicating a restriction so that only terms with positive interaction are taken into account, and similarly for I^-, J^+ and J^-.

Looking at the expression of the Choquet integral w.r.t the Möbius transform, we recognize the Lovász extension of v. This establish the link with pseudo-Boolean functions. It can be seen also as an extension of fuzzy measures (or set functions) to fuzzy sets (which are defined by a membership function).

8 k-additive measures

k-order additive measures or *k-additive measures* for short have been introduced by Grabisch in an attempt to decrease the exponential complexity of fuzzy measures in practical applications, since a fuzzy measure defined on a set of n elements requires 2^n real coefficients for its definition. A means which has been often used for this is to introduce the property of *decomposability*: a fuzzy measure μ is decomposable if the measure of any subset can be expressed as a function of the measures of each element in the set. Thus we need only to define the *distribution* of μ over N, hence n coefficients instead of 2^n. The most usual example in this category are *additive* measures. But it appears that this is too drastic a simplification, which is too limitative, especially in multiattribute decision making. One can think of a distribution defined not only for singletons, but also for pairs. Nevertheless, how to define properly the measure of triples, etc. only with the help of the distribution?

The answer to this question was given (again!) by looking at pseudo-Boolean functions, and especially their multilinear form (11). An additive measure (defined by a distribution on singletons) has a linear expression $f(x) = \sum_i a_i x_i$, and the coefficients a_i's (i.e. the Möbius transform) are indeed the distribution itself. Then a fuzzy measure of which the multilinear extension is limited to a degree 2, (or 3, etc.) can be expressed only by a distribution on singletons and pairs (or also on triples, etc.). Equation (11) will then provide the value of μ for evey subset.

Definition 2 *A fuzzy measure μ is said to be k-*additive *if its Möbius transform satisfies $m(S) = 0$ for any S such that $s > k$, and there exists at least one subset S of N of exactly k elements such that $m(S) \neq 0$.*

The following property of k-additive measures is fundamental.

Property 12 *Let μ be a k-additive measure on N. Then*

(i) $I(S) = \check{m}(S) = 0$ *for every $S \subset N$ such that $|S| > k$,*

(ii) $I(S) = J(S) = m(S) = \check{\mu}(S)$ *for every $S \subset N$ such that $|S| = k$,*

(iii) if μ is a 2-additive measure, $I(S) = J(S)$ *for every $S \subset N$.*

Thus, k-additive measures can be represented by a limited set of coefficients, either $m(S)$, $|S| \leq k$, or $I(S)$, $|S| \leq k$, or equivalently $\check{\mu}(S)$, $J(S)$, $|S| \leq k$ i.e. at most $\sum_{i=1}^{k} \binom{n}{i}$ coefficients.

Of course, we can think of extending the definition to any set function. A k-additive set function (although the term seems to be improper) has a multilinear expression of degree k.

As said in the introduction, the concept of k-additivity in a sense generalizes the concept of decomposability, but only for the case where decomposability is understood as additivity. It is then tempting to define k-*decomposability* in a similar way. This amounts to redefine the Möbius transform in an adequate way. Using pseudo-addition operations for the decomposability, Mesiar [23] has proposed what he called *k-order pan-additive measures*. Based on this and on works of Marichal et al. [22], Grabisch has completed the picture by giving the corresponding Shapley interaction index [12]. It is interesting here to focus on the case of the maximum operator for the pseudo-addition, the only case which is really problematic. As a fuzzy measure which is decomposable w.r.t the maximum operator is called a *possibility measure*, it is natural here to speak of *k-possibility measures*. We sketch briefly the definition and main properties of them (see [12] for more details).

For any fuzzy measure μ, its $\vee$-Möbius transform is defined by

$$m^{\mu}_{\vee}(A) := \bigvee_{\substack{B \subset A \\ |A \setminus B| \text{ even}}} \mu(B) \neg_{\vee} \bigvee_{\substack{B \subset A \\ |A \setminus B| \text{ odd}}} \mu(B). \tag{35}$$

where the residuated difference $\neg_{\vee}$ is defined by

$$x \neg_{\vee} y := \begin{cases} x, & \text{if } x > y \\ 0, & \text{otherwise.} \end{cases}$$

Note that this Möbius transform assumes only positive values.

A *k-possibility measure*, denoted ${}^{k}\Pi$, is any fuzzy measure such that its $\vee$-Möbius transform vanishes for subsets of more than k elements. It can be shown that a simple expression of k-possibility measures is

$${}^{k}\Pi(A) = \bigvee_{B \subset A, |B| = \min(|A|, k)} {}^{k}\Pi(B). \tag{36}$$

Observe that if $|A| \geq k$, then ${}^{k}\Pi(A)$ is obtained as the supremum over all B of k elements included in A. If $|A| < k$, then we get simply ${}^{k}\Pi(A)$. Thus, as in the additive case, ${}^{k}\Pi$ is determined by the value of the set function over all subsets of at most k elements.

We can define *k-necessity measures* by duality, i.e.

$${}^{k}\mathrm{N}(A) = 1 - {}^{k}\Pi(A^c) = \bigwedge_{B \in A^c, |B| = \min(|A|, k)} (1 - {}^{k}\Pi(B)). \tag{37}$$

We can show the following properties.

Proposition 3 *Let ${}^{k}\Pi$ be a k-possibility measure. The following properties hold.*

(i) ${}^{k}\Pi(A \cup B) \geq {}^{k}\Pi(A) \vee {}^{k}\Pi(B)$*, for any* $A, B \subset X$

(ii) ${}^{k}\Pi$ *is a submodular measure, i.e.*

$${}^{k}\Pi(A \cup B) + {}^{k}\Pi(A \cap B) \leq {}^{k}\Pi(A) + {}^{k}\Pi(B).$$

Dual properties on ${}^{k}\mathrm{N}$ can be obtained as well.

9 Approximation of set functions and fuzzy measures

The section on k-additive measures suggests the following question, which can be asked for set functions as well.

Since k-additive measures are much simpler to manipulate, one can think of replacing a given fuzzy measure by a k-additive one, such that a loss of information as low as possible. How to choose the best (in some sense) k-additive measure? How good is the approximation?

Obviously, the greater k, the better the approximation, but the quality should depend also on the type of fuzzy measure.

In [19], Hammer and Holzman have addressed this question, considering a squared error criterion computed over all the vertices of the hypercube $[0,1]^n$. For a given pseudo-Boolean function f, let us consider its best kth approximation $f^{(k)}$, which minimizes the following criterion:

$$\sum_{x\in\{0,1\}^n} [f(x) - f^k(x)]^2 \tag{38}$$

among all multilinear polynomials of degree at most k. They proved that the best approximation is given by the unique solution $\{a^{(k)}(T) | T \subset N, |T| \leq k\}$ of the triangular linear system:

$$\frac{1}{2^n}\sum_{x\in\{0,1\}^n} \Delta_S f^{(k)}(x) = \frac{1}{2^n}\sum_{x\in\{0,1\}^n} \Delta_S f(x), \quad \forall S \subset N, \;\; s \leq k.$$

They solved the system for $k = 1$ and $k = 2$. In [16], the general solution is given:

$$a^{(k)}(S) = a(S) + (-1)^{k-s} \sum_{\substack{T\supset S\\ t>k}} \binom{t-s-1}{k-s} \frac{1}{2^{t-s}} a(T), \quad \forall S \subset N, \;\; s \leq k. \tag{39}$$

For $k = 1$, the solution reduces to:

$$a^{(1)}(\emptyset) = \sum_{T\subset N} \frac{-(t-1)}{2^t} a(T) \tag{40}$$

$$a^{(1)}(\{i\}) = \sum_{T\ni i} \frac{1}{2^{t-1}} a(T), \quad \forall i \in N, \tag{41}$$

and for $k = 2$:

$$a^{(2)}(\emptyset) = \sum_{T\subset N} \frac{(t-1)(t-2)}{2^{t+1}} a(T) \tag{42}$$

$$a^{(2)}(\{i\}) = \sum_{T\ni i} \frac{-(t-2)}{2^{t-1}} a(T), \quad \forall i \in N \tag{43}$$

$$a^{(2)}(\{i,j\}) = \sum_{T\ni i,j} \frac{1}{2^{t-2}} a(T), \quad \forall i,j \in N. \tag{44}$$

Observe that for $k = 1$, we see that $J(\{i\})$ is the solution for $a^{(1)}(\{i\})$, while for $k = 2$, $J(\{i,j\})$ is solution for $a^{(2)}(\{i,j\})$. This shows the close relation of the Banzhaf interaction with this kind of approximation.

When $k = 1$, it has been shown that one can recover the Shapley index as well, if the criterion is a weighted one:

$$\text{minimize} \sum_{S \subset N, S \neq N} \frac{1}{\binom{n-2}{s-1}} [f(1_S) - f^{(1)}(1_S)]^2. \tag{45}$$

Another way to consider the approximation problem is to consider upper and lower envelopes, which has a probabilistic flavour. We consider here only fuzzy measures, and for a given fuzzy measure μ and a given k, we want to obtain the set of k-additive measures which dominate μ (upper approximation), or are dominated by μ (lower approximation). A measure ν *dominates* a measure μ if for all $A \subset N$, we have always $\nu(A) \geq \mu(A)$.

This problem has been addressed in [13, 15], and we give here the main result, which generalizes a result of Chateauneuf and Jaffray [3] obtained for $k = 1$ (probability measure).

Theorem 1 *Let μ be a fuzzy measure on N, m its Möbius transform, and suppose that μ^* is a k-additive measure which dominates μ, $1 \leq k \leq n-1$. Then necessarily, the Möbius transform m^* of μ^* can be put under the following form:*

$$m^*(A) = \sum_{B \cap A \neq \emptyset} \lambda(B, A) m(B), \forall A \in {}^k\mathcal{P}(N), \tag{46}$$

where ${}^k\mathcal{P}(N)$ denotes the power set limited to subsets of at most k elements. Moreover, the weight function $\lambda : \mathcal{P}(N) \times {}^k\mathcal{P}(N) \longrightarrow$ is such that

$$\sum_{A | A \cap B \neq \emptyset} \lambda(B, A) = 1, \forall B \subset N, \tag{47}$$

$$\lambda(B, A) = 0, \forall A, A \cap B = \emptyset. \tag{48}$$

The theorem gives only a necessary condition, thus any k-additive measure built by this process is not necessarily a dominating measure (anyway, not all fuzzy measures can be dominated by a k-additive measure). When $k = 1$, it is known that, if the fuzzy measure can be dominated (e.g. in the case of convex fuzzy measures), the Shapley value is a dominating additive measure, and moreover, it is the center of gravity of the convex set of dominating additive measures.

References

[1] J.F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

[2] C. Berge. *Principles of Combinatorics*. Academic Press, 1971.

[3] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[4] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[5] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.*, 38:325–339, 1967.

[6] D. Denneberg. *Non-Additive Measure and Integral*. Kluwer Academic, 1994.

[7] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, to appear.

[8] M. Grabisch. k-order additive fuzzy measures. In *6th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU)*, pages 1345–1350, Granada, Spain, July 1996.

[9] M. Grabisch. Alternative expressions of the discrete Choquet integral. In *7th IFSA World Congress*, Prague, Czech Republic, June 1997.

[10] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[11] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[12] M. Grabisch. On the representation of k-decomposable measures. In *7th IFSA World Congress*, Prague, Czech Republic, June 1997.

[13] M. Grabisch. k-additive measures: Recent issues and challenges. In *5th Int. Conf. on Soft Computing and Information/Intelligent Systems*, Iizuka, Japan, October 1998.

[14] M. Grabisch. On lower and upper approximation of fuzzy measures by k-order additive measures. In *7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, pages 1577–1584, Paris, France, July 1998.

[15] M. Grabisch. On upper approximation of fuzzy measures by k-order additive measures. In B. Bouchon-Meunier, R.R. Yager, and L. Zadeh, editors, *Information, Uncertainty, Fusion*. Kluwer Scientific Publ., to appear.

[16] M. Grabisch, J.L. Marichal, and M. Roubens. Equivalent representations of a set function with applications to game theory and multicriteria decision making. working paper.

[17] M. Grabisch and M. Roubens. Equivalent representations of a set function with application to decision making. In *6th IEEE Int. Conf. on Fuzzy Systems*, Barcelona, Spain, July 1996.

[18] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, submitted.

[19] P.L. Hammer and R. Holzman. On approximations of pseudo-Boolean functions. *ZOR - Methods and Models of Operations Research*, 36:3–21, 1992.

[20] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas*. Springer, 1968.

[21] L. Lovász. Submodular function and convexity. In A. Bachem, M. Grötschel, and B. Korte, editors, *Mathematical programming. The state of the art*, pages 235–257. Springer Verlag, 1983.

[22] J.L. Marichal, P. Mathonet, and E. Tousset. Mesures floues définies sur une échelle ordinale. working paper, 1996.

[23] R. Mesiar. k-order pan-additive discrete fuzzy measures. In *7th IFSA World Congress*, pages 488–490, Prague, Czech Republic, June 1997.

[24] P. Miranda and M. Grabisch. Optimization issues for fuzzy measures. In *7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, pages 1204–1211, Paris, France, July 1998.

[25] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[26] G. Owen. Multilinear extensions of games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 139–151. Cambridge University Press, 1988.

[27] E. Pap. *Null-Additive Set Functions*. Kluwer Academic, 1995.

[28] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[29] M. Roubens. Interaction between criteria and definition of weights in MCDA problems. In *44th Meeting of the European Working Group "Multicriteria Aid for Decisions"*, Brussels, Belgium, October 1996.

[30] G. Shafer. *A Mathematical Theory of Evidence*. Princeton Univ. Press, 1976.

[31] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[32] I. Singer. Extensions of functions of 0-1 variables and applications to combinatorial optimization. *Numerical Functional Analysis and Optimization*, 7(1):23–62, 1984.

[33] P. Walley. Coherent lower (and upper) probabilities. Technical Report 22, University of Warvick, Coventry, 1981.

[34] Z. Wang and G.J. Klir. *Fuzzy measure theory*. Plenum, 1992.