A nonparametric test of exchangeability for extreme-value and left-tail decreasing bivariate copulas

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Abstract

A nonparametric rank-based test of exchangeability for bivariate extreme-value copulas is first proposed. The two key ingredients of the suggested approach are the nonparametric rank-based estimators of the Pickands dependence function recently studied by Genest and Segers, and a multiplier technique for obtaining approximate p-values for the derived statistics. The proposed approach is then extended to left-tail decreasing dependence structures that are not necessarily extreme-value copulas. Large-scale Monte Carlo experiments are used to investigate the level and power of the various versions of the test and show that the proposed procedure can be substantially more powerful than tests of exchangeability derived directly from the empirical copula. The approach is illustrated on well-known financial data.

Keywords: multiplier central limit theorem, nonparametric estimation, Pickands dependence function, ranks.

Running head: A test of exchangeability for EV and LTD copulas.

1 Introduction

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a random sample from an unknown bivariate distribution H with unknown continuous margins F and G. To obtain a parametric model for H, a frequently used starting point in many applications is Sklar (1959)'s representation theorem, which states that H can be expressed as

$$H(x, y) = C\{F(x), G(y)\}, \qquad x, y \in \mathbb{R},$$

in terms of a unique bivariate copula C.

A class of copulas of increasing practical interest in fields such as finance, hydrology and insurance is that of *extreme-value* copulas. These copulas arise in the limiting joint distribution of suitably normalized componentwise maxima in random samples (see e.g. Galambos, 1987; McNeil et al., 2005; Gudendorf and Segers, 2010) which makes them natural tools for modeling the dependence between extreme observations. In the bivariate case, extreme-value copulas are characterized by a convex function $A : [0, 1] \to [1/2, 1]$ satisfying $\max(t, 1 - t) \le A(t) \le 1$ for all $t \in [0, 1]$, and can be represented as

$$C(u, v) = \exp\left[\log(uv)A\left\{\frac{\log(v)}{\log(uv)}\right\}\right], \qquad (u, v) \in (0, 1]^2 \setminus \{(1, 1)\}. \tag{1}$$

The function A is commonly referred to as the *Pickands dependence function* (Pickands, 1981).

Statistical inference procedures for bivariate extreme-value copulas are progressively being developed: Ghoudi et al. (1998), Ben Ghorbal et al. (2009) and Kojadinovic and Yan (2010b) have proposed tests of extreme-value dependence, Genest and Segers (2009) have studied nonparametric estimators of the Pickands dependence function, and Genest et al. (2011a) have addressed the goodness-of-fit issue. The aim of this work is to develop a nonparametric test of exchangeability for extreme-value copulas, i.e., a test that can be used to assess whether the unknown extreme-value copula C satisfies $C(u, v) = C(v, u)$ for all $u, v \in [0, 1]$. In the copula-modeling context, such a test can typically be used to decrease the number of candidate copula families by eliminating for instance those that are asymmetric if exchangeability is not rejected.

When dealing with extreme-value copulas, it follows from (1) that exchangeability is equivalent to having $A(t) = A(1 - t)$ for all $t \in [0, 1]$. Given a nonparametric estimator A_n of the unknown Pickands dependence function, it seems natural to base tests of exchangeability for extreme-value copulas on the process

$$\sqrt{n}\,\{A_n(t) - A_n(1 - t)\}, \qquad t \in [0, 1]. \tag{2}$$

The first key ingredient of the proposed approach are the nonparametric estimators of A studied by Genest and Segers (2009). These estimators depend on the data only through the pairs of normalized ranks

$$U_{i,n} = \frac{1}{n+1}\sum_{j=1}^{n} \mathbf{1}(X_j \le X_i), \qquad V_{i,n} = \frac{1}{n+1}\sum_{j=1}^{n} \mathbf{1}(Y_j \le Y_i), \qquad i \in \{1, \ldots, n\},$$

which ensures that the studied inference procedure is margin-free. As the weak limit of the resulting version of process (2) turns out to be unwieldy, a multiplier technique is used to compute approximate p-values for statistics derived from (2). This is the second key ingredient of the proposed approach and is based on the seminal work of Scaillet (2005) and Rémillard and Scaillet (2009), recently thoroughly revisited by Segers (2011).

An interesting consequence of the present study is that the multiplier central limit theorems obtained in Sections 3 and 4 to justify the proposed approach can be used to show the asymptotic validity of most nonparametric multiplier-based tests involving the rank-based estimators of A studied by Genest and Segers (2009).

The paper is organized as follows. In the second section, recent results on the nonparametric rank-based estimation of the Pickands dependence function are reviewed. The proposed test of exchangeability for extreme-value copulas is then described in full detail in the third section. An extension of the test to left-tail decreasing copulas is studied in the fourth section which significantly broadens the applicability of the proposed approach. The finite-sample performance of the various versions of the test is then investigated in a large-scale Monte Carlo experiment whose results are partially reported in Section 5. The obtained rejection rates are also compared with those of the tests of exchangeability studied by Genest et al. (2011b) which are not specific to extreme-value or left-tail decreasing copulas. An illustration is presented in the last section and concluding remarks are given. All the proofs and the full simulation results are relegated to the section "Supporting material" of the online version of the paper.

The following notational conventions are used in the paper. For any $x, y \in \mathbb{R}$, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. Furthermore, $\ell^\infty([0,1]^2)$ represents the space of bounded real-valued functions on $[0,1]^2$, while $\mathcal{C}([0,1])$ represents the space of continuous real-valued functions on $[0,1]$; both are equipped with the uniform metric. The arrow $\leadsto$ denotes weak convergence in the sense of van der Vaart and Wellner (2000) while $\overset{\Pr}{\to}$ denotes convergence in probability.

Finally, note that all the tests studied in this work are implemented in the R package `copula` (Kojadinovic and Yan, 2010a) available on the Comprehensive R Archive Network.

2 Nonparametric rank-based estimation of A

2.1 The Pickands and Capéraà–Fougères–Genest estimators

Two nonparametric estimators of the unknown Pickands dependence function A were recently studied by Genest and Segers (2009). They are the rank-based versions of two well-known estimators of A, namely the Pickands estimator (Pickands, 1981) and the Capéraà–Fougères–Genest estimator (Capéraà et al., 1997). The latter will be abbreviated as CFG in the sequel.

Let

$$S_{i,n} = -\log(U_{i,n}) \qquad \text{and} \qquad T_{i,n} = -\log(V_{i,n}),$$

for every $i \in \{1, \dots, n\}$, and let

$$\xi_{i,n}(0) = S_{i,n}, \quad \xi_{i,n}(1) = T_{i,n}, \quad \text{and} \quad \xi_{i,n}(t) = \left(\frac{S_{i,n}}{1-t}\right) \wedge \left(\frac{T_{i,n}}{t}\right),$$

for every $i \in \{1, \dots, n\}$ and any $t \in (0,1)$. The rank-based version of the Pickands and CFG estimators are then respectively defined by

$$A_n^{\mathrm{P}}(t) = 1 \Big/ \frac{1}{n} \sum_{i=1}^n \xi_{i,n}(t), \qquad \text{and} \qquad A_n^{\mathrm{CFG}}(t) = \exp\left[-\gamma - \frac{1}{n} \sum_{i=1}^n \log \xi_{i,n}(t)\right], \quad t \in [0,1],$$

where $\gamma = -\int_0^\infty \log(x) e^{-x} \mathrm{d}x \approx 0.577$ is Euler's constant. The previous estimators can be expressed in terms of the empirical copula

$$\hat{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(U_{i,n} \leq u, V_{i,n} \leq v), \qquad u, v \in [0,1],$$

as

$$A_n^{\mathrm{P}}(t) = 1 \Big/ \int_0^1 \hat{C}_n(x^{1-t}, x^t) \frac{\mathrm{d}x}{x},$$

and

$$A_n^{\mathrm{CFG}}(t) = \exp\left[-\gamma + \int_0^1 \left\{ \hat{C}_n(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1}) \right\} \frac{\mathrm{d}x}{x \log x} \right].$$

The estimators being rank-based, we have $A_n^{\mathrm{P}}(0) = A_n^{\mathrm{P}}(1)$ and $A_n^{\mathrm{CFG}}(0) = A_n^{\mathrm{CFG}}(1)$. To ensure that the endpoint constraints $A_n^{\mathrm{P}}(0) = A_n^{\mathrm{CFG}}(0) = 1$ and $A_n^{\mathrm{P}}(1) = A_n^{\mathrm{CFG}}(1) = 1$ are satisfied, the previous estimators can be corrected as suggested in Deheuvels (1991) for A_n^{P} and as in Capéraà et al. (1997) for A_n^{CFG}, which yields the corrected versions

$$1/A_{n,c}^{\mathrm{P}}(t) = 1/A_n^{\mathrm{P}}(t) - (1-t)\{1/A_n^{\mathrm{P}}(0) - 1\} - t\{1/A_n^{\mathrm{P}}(1) - 1\} = 1/A_n^{\mathrm{P}}(t) - 1/A_n^{\mathrm{P}}(0) + 1,$$

and

$$\log A_{n,c}^{\mathrm{CFG}}(t) = \log A_n^{\mathrm{CFG}}(t) - (1-t)\log A_n^{\mathrm{CFG}}(0) - t \log A_n^{\mathrm{CFG}}(1) = \log A_n^{\mathrm{CFG}}(t) - \log A_n^{\mathrm{CFG}}(0),$$

respectively. The above corrected versions were found to behave better than the uncorrected versions in small samples in Genest and Segers (2009) and Genest et al. (2011a). Notice that A_n^{P} and $A_{n,c}^{\mathrm{P}}$ (resp. A_n^{CFG} and $A_{n,c}^{\mathrm{CFG}}$) become indistinguishable as n tends to infinity.

2.2 The empirical copula process

The limiting behavior of $\mathbb{A}_n^{\mathrm{P}} = \sqrt{n}(A_n^{\mathrm{P}} - A)$ and $\mathbb{A}_n^{\mathrm{CFG}} = \sqrt{n}(A_n^{\mathrm{CFG}} - A)$ was established in Theorem 3.2 of Genest and Segers (2009) and depends on the weak limit $\mathbb{C}$ of the empirical copula process $\sqrt{n}(\hat{C}_n - C)$. To present the recent fundamental results obtained by Segers (2011) on this subject in their utmost generality, let us assume, for a short while, that the copula C of the random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ is not necessarily an extreme-value copula.

Let

$$C^{(1)}(u,v) = \lim_{h \to 0} \frac{C(u+h, v) - C(u,v)}{h}, \qquad u \in (0,1), v \in [0,1],$$

and

$$C^{(2)}(u,v) = \lim_{h \to 0} \frac{C(u, v+h) - C(u,v)}{h}, \qquad u \in [0,1], v \in (0,1),$$

be the first-order partial derivatives of C, assuming they exist. As suggested by Segers (2011), the domain of $C^{(1)}$ and $C^{(2)}$ can be extended to the whole of $[0,1]^2$ by setting, for any $v \in [0,1]$,

$$C^{(1)}(0,v) = \limsup_{h \downarrow 0} \frac{C(h,v)}{h}, \qquad C^{(1)}(1,v) = \limsup_{h \downarrow 0} \frac{v - C(1-h, v)}{h},$$

and, for any $u \in [0,1]$,

$$C^{(2)}(u,0) = \limsup_{h \downarrow 0} \frac{C(u,h)}{h}, \qquad C^{(2)}(u,1) = \limsup_{h \downarrow 0} \frac{u - C(u, 1-h)}{h}.$$

In many cases, $C^{(1)}$ and $C^{(2)}$ fail to be continuous on the whole of $[0,1]^2$. Segers (2011) showed, for instance, that the first-order partial derivatives of extreme-value copulas are not continuous at $(1,1)$ with the exception of the independence copula, and that for many extreme-value copulas, such as the Gumbel-Hougaard, $C^{(1)}$ and $C^{(2)}$ are not continuous at $(0,0)$. However, for many copula families, including extreme-value copulas, $C^{(1)}$ is continuous on $(0,1)\times[0,1]$, and $C^{(2)}$ is continuous on $[0,1]\times(0,1)$. Under these less restrictive continuity conditions, Segers (2011) showed that the original result of Gänssler and Stute (1987) (derived under the assumption of continuity of the partial derivatives on $[0,1]^2$) holds, i.e., that $\sqrt{n}(\hat{C}_n - C)$ converges weakly to

$$\mathbb{C}(u,v) = \alpha(u,v) - C^{(1)}(u,v)\alpha(u,1) - C^{(2)}(u,v)\alpha(1,v), \qquad u,v \in [0,1], \tag{3}$$

in $\ell^\infty([0,1]^2)$, where α is a C-Brownian bridge, i.e., a tight centered Gaussian process on $[0,1]^2$ with $E[\alpha(u,v)\alpha(u',v')] = C(u\wedge u', v\wedge v') - C(u,v)C(u',v')$, $u,v,u',v' \in [0,1]$.

2.3 Back to the limiting behavior $\mathbb{A}_n^{\mathrm{P}}$ and $\mathbb{A}_n^{\mathrm{CFG}}$

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be random sample from c.d.f. $C\{F(x),G(y)\}$, where C is an extreme-value copula whose associated Pickands dependence function A is twice continuously differentiable on $(0,1)$ with $\sup_{t\in(0,1)}\{t(1-t)A''(t)\} < \infty$. Then, from Theorem 3.2 of Genest and Segers (2009) (see also Segers, 2011, Example 5.3), we have that

$$\mathbb{A}_n^{\mathrm{P}}(t) = \sqrt{n}\{A_n^{\mathrm{P}}(t) - A(t)\} \rightsquigarrow \mathbb{A}^{\mathrm{P}}(t) = -A^2(t)\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}, \tag{4}$$

and

$$\mathbb{A}_n^{\mathrm{CFG}}(t) = \sqrt{n}\{A_n^{\mathrm{CFG}}(t) - A(t)\} \rightsquigarrow \mathbb{A}^{\mathrm{CFG}}(t) = A(t)\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \tag{5}$$

in $\mathcal{C}([0,1])$.

3 Description of the test

As discussed in the introduction, testing the exchangeability of an extreme-value copula amounts to testing the symmetry of its Pickands dependence function with respect to the axis $t = 1/2$. It therefore seems natural to base a test of exchangeability on (2), with A_n being either $A_{n,c}^{\mathrm{P}}$ or $A_{n,c}^{\mathrm{CFG}}$.

The following proposition, which immediately follows from (4), (5), and the continuous mapping theorem, gives the weak limit of the test process (2) under the null hypothesis $H_0 : A(t) = A(1-t)$ for all $t \in [0,1]$.

Proposition 1. *Under H_0, we have that*

$$\sqrt{n}\,\{A_n(t) - A_n(1-t)\} \rightsquigarrow \mathbb{A}(t) - \mathbb{A}(1-t)$$

in $\mathcal{C}([0,1])$, where $\mathbb{A}$ is either $\mathbb{A}^{\mathrm{P}}$ or $\mathbb{A}^{\mathrm{CFG}}$.

As the previous weak limit is unwieldy, to obtain approximate p-values for statistics derived from (2), we adapt to the current setting the multiplier approach used by Rémillard and Scaillet (2009) and revisited in Segers (2011). It consists of approximating the weak limit of the test process using an approximation of the weak limit $\mathbb{C}$ of the empirical copula process on which it depends.

In order to approximate $\mathbb{C}$ in the spirit of Rémillard and Scaillet (2009), it is necessary to estimate the partial derivatives of C. From (4) and (5), and the expression of $\mathbb{C}$ given in (3), these partial derivatives are needed at points of the form (x^{1-t}, x^t). Under the assumption that A is differentiable on $[0, 1]$ with derivative A', and starting from (1), for any $x \in (0, 1)$ and $t \in [0, 1]$, one obtains

$$C^{(1)}(x^{1-t}, x^t) = \{A(t) - tA'(t)\}x^{A(t)-(1-t)},$$

and

$$C^{(2)}(x^{1-t}, x^t) = \{A(t) + (1-t)A'(t)\}x^{A(t)-t}.$$

One natural way of estimating the desired quantities then consists of plugging estimators of A and A' in the previous expressions. Let A_n be either $A^{\mathrm{P}}_{n,c}$ or $A^{\mathrm{CFG}}_{n,c}$. The estimator of A that we consider is then

$$\hat{A}_n = (A_n \wedge 1) \vee I \vee (1 - I),$$

where I is the identity function. The previous definition ensures that $t \vee (1-t) \le \hat{A}_n(t) \le 1$ for all $t \in [0, 1]$, a property that will be used to verify that the testing procedure is asymptotically valid. From (4) and (5), we have that $\sup_{t\in[0,1]} |A_n(t) - A(t)| \overset{\Pr}{\to} 0$, which implies that $\sup_{t\in[0,1]} |\hat{A}_n(t) - A(t)| \overset{\Pr}{\to} 0$ since $|\hat{A}_n(t) - A(t)| \le |A_n(t) - A(t)|$ for all $t \in [0, 1]$ by construction.

Starting from A_n and by analogy with the definition adopted in Segers (2010, page 9), a consistent estimator of A' can be defined as

$$A'_n(t) = \frac{1}{2n^{-1/2}} \begin{cases} \{A_n(2n^{-1/2}) - A_n(0)\}, & \text{if } t \in [0, n^{-1/2}), \\ \{A_n(t + n^{-1/2}) - A_n(t - n^{-1/2})\}, & \text{if } t \in [n^{-1/2}, 1 - n^{-1/2}], \\ \{A_n(1) - A_n(1 - 2n^{-1/2})\}, & \text{if } t \in (1 - n^{-1/2}, 1]. \end{cases} \tag{6}$$

A proof of the following result is available in the section "Supporting material" of the online version of the paper.

Proposition 2. *Suppose that A is twice continuously differentiable on $(0, 1)$ with $\sup_{t\in(0,1)}\{t(1-t)A''(t)\} < \infty$. Then, $\sup_{t\in[0,1]} |A'_n(t) - A'(t)| \overset{\Pr}{\to} 0$.*

Consistent estimators of the quantities $C^{(1)}(x^{1-t}, x^t)$ and $C^{(2)}(x^{1-t}, x^t)$, $x \in (0, 1)$, $t \in [0, 1]$, are therefore given by

$$\hat{C}^{(1)}_{A_n}(x^{1-t}, x^t) = \{\hat{A}_n(t) - tA'_n(t)\}x^{\hat{A}_n(t)-(1-t)}, \tag{7}$$

and

$$\hat{C}^{(2)}_{A_n}(x^{1-t}, x^t) = \{\hat{A}_n(t) + (1-t)A'_n(t)\}x^{\hat{A}_n(t)-t}. \tag{8}$$

Before stating the key result that provides an asymptotic justification to the adopted approach, let us first introduce additional notation. Let N be a large integer, and let $Z_{i,k}$, $i \in \{1,\dots,n\}$, $k \in \{1,\dots,N\}$, be i.i.d. random variables with mean 0, variance 1, satisfying $\int_0^\infty \{\Pr(|Z_{i,k}| > x)\}^{1/2} \mathrm{d}x < \infty$, and independent of the data $(X_1,Y_1),\dots,(X_n,Y_n)$. Furthermore, for any $k \in \{1,\dots,N\}$, let

$$\begin{aligned}\hat\alpha_{n,k}(u,v) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_{i,k} \left\{ \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v) - \hat C_n(u,v) \right\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_{i,k} - \bar Z_k) \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v), \qquad u,v \in [0,1],\end{aligned} \tag{9}$$

where $\bar Z_k = n^{-1} \sum_{i=1}^n Z_{i,k}$. Finally, for any $k \in \{1,\dots,N\}$, any $x \in (0,1)$, and any $t \in [0,1]$, let

$$\hat{\mathbb{C}}_{n,k}(x^{1-t},x^t) = \hat\alpha_{n,k}(x^{1-t},x^t) - \hat C_{A_n}^{(1)}(x^{1-t},x^t)\hat\alpha_{n,k}(x^{1-t},1) - \hat C_{A_n}^{(2)}(x^{1-t},x^t)\hat\alpha_{n,k}(1,x^t), \tag{10}$$

be an approximate independent copy of $\mathbb{C}$ evaluated at (x^{1-t},x^t), let

$$\hat{\mathbb{A}}^{\mathrm{P}}_{n,k}(t) = -\{A^{\mathrm{P}}_{n,c}(t)\}^2 \int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t},x^t) \frac{\mathrm{d}x}{x}, \tag{11}$$

and let

$$\hat{\mathbb{A}}^{\mathrm{CFG}}_{n,k}(t) = A^{\mathrm{CFG}}_{n,c}(t) \int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t},x^t) \frac{\mathrm{d}x}{x \log x}. \tag{12}$$

Note that the integrals appearing in the two previous expressions are not indefinite. Indeed, as can be verified from (9), $\hat\alpha_{n,k}(u,v) = 0$ if $u \wedge v < 1/(n+1)$ or if $n/(n+1) \le u \wedge v$. Since $x^{1-t} \wedge x^t$ tends to zero (resp. one) when x tends to zero (resp. one), we see that $\hat{\mathbb{C}}_{n,k}(x^{1-t},x^t)$ becomes equal to zero for x sufficiently close to zero or one. Notice also that $\hat{\mathbb{C}}_{n,k}(x,1) = \hat{\mathbb{C}}_{n,k}(1,x) = 0$ for all $x \in (0,1)$, which implies that $\hat{\mathbb{A}}^{\mathrm{P}}_{n,k}(0) = \hat{\mathbb{A}}^{\mathrm{CFG}}_{n,k}(0) = \hat{\mathbb{A}}^{\mathrm{P}}_{n,k}(1) = \hat{\mathbb{A}}^{\mathrm{CFG}}_{n,k}(1) = 0$ as expected.

The following result, whose proof is available in the section "Supporting material" of the online version of the paper, is at the root of the proposed test of exchangeability.

Theorem 1. *Let $(X_1,Y_1),\dots,(X_n,Y_n)$ be a random sample from c.d.f. $C\{F(x),G(y)\}$ where C is an extreme-value copula whose associated Pickands dependence function A is twice continuously differentiable on $(0,1)$ with $\sup_{t\in(0,1)}\{t(1-t)A''(t)\} < \infty$. Then,*

$$\left(\mathbb{A}^{\mathrm{P}}_n, \hat{\mathbb{A}}^{\mathrm{P}}_{n,1}, \dots, \hat{\mathbb{A}}^{\mathrm{P}}_{n,N}\right) \rightsquigarrow \left(\mathbb{A}^{\mathrm{P}}, \mathbb{A}^{\mathrm{P}}_1, \dots, \mathbb{A}^{\mathrm{P}}_N\right) \tag{13}$$

and

$$\left(\mathbb{A}^{\mathrm{CFG}}_n, \hat{\mathbb{A}}^{\mathrm{CFG}}_{n,1}, \dots, \hat{\mathbb{A}}^{\mathrm{CFG}}_{n,N}\right) \rightsquigarrow \left(\mathbb{A}^{\mathrm{CFG}}, \mathbb{A}^{\mathrm{CFG}}_1, \dots, \mathbb{A}^{\mathrm{CFG}}_N\right) \tag{14}$$

in $\mathcal{C}([0,1])^{\otimes(N+1)}$, where $\mathbb{A}^{\mathrm{P}}_1,\dots,\mathbb{A}^{\mathrm{P}}_N$ are independent copies of $\mathbb{A}^{\mathrm{P}}$, and $\mathbb{A}^{\mathrm{CFG}}_1,\dots,\mathbb{A}^{\mathrm{CFG}}_N$ are independent copies of $\mathbb{A}^{\mathrm{CFG}}$.

A natural candidate test statistic is then the Cramér–von Mises functional

$$S_n = \int_0^{\frac{1}{2}} n \left\{ A_n(t) - A_n(1-t) \right\}^2 \mathrm{d}t. \tag{15}$$

Next, for any $k \in \{1, \ldots, N\}$, let

$$S_{n,k} = \int_0^{\frac{1}{2}} \left\{ \hat{\mathbb{A}}_{n,k}(t) - \hat{\mathbb{A}}_{n,k}(1-t) \right\}^2 \mathrm{d}t,$$

where $\hat{\mathbb{A}}_{n,k}$ stands for either $\hat{\mathbb{A}}^{\mathrm{P}}_{n,k}$ or $\hat{\mathbb{A}}^{\mathrm{CFG}}_{n,k}$. From Theorem 1 and the continuous mapping theorem, we immediately have that, under H_0,

$$(S_n, S_{n,1}, \ldots, S_{n,N}) \rightsquigarrow (S, S_1, \ldots, S_N)$$

in $[0, \infty)^{\otimes (N+1)}$, where S is the weak limit of S_n, and $S_1, \ldots, S_N$ are independent copies of S. This suggests computing an approximate p-value for S_n as

$$\frac{1}{N} \sum_{k=1}^{N} \mathbf{1} \left(S_{n,k} \geq S_n \right).$$

From a practical perspective, to compute the statistic S_n and the $S_{n,k}$, $k \in \{1, \ldots, N\}$, we used a fine grid of m uniformly spaced points on $(0, 1/2)$ (m was set to 100 in the Monte Carlo experiments of Section 5). The computation of the $S_{n,k}$, $k \in \{1, \ldots, N\}$, requires in turn the computation of the integrals appearing in (11) and (12). The following result, whose proof is available in the section "Supporting material" of the online version of the paper, shows that the integral appearing in (11) can be easily computed.

Proposition 3. *For any $k \in \{1, \ldots, N\}$ and any $t \in (0, 1)$, there holds*

$$\int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_{i,k} - \bar{Z}_k) \left[\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t} \right.$$
$$\left. - \frac{\hat{A}_n(t) - tA_n'(t)}{\hat{A}_n(t) - (1-t)} \left\{ 1 - U_{i,n}^{\frac{\hat{A}_n(t)-(1-t)}{1-t}} \right\} - \frac{\hat{A}_n(t) + (1-t)A_n'(t)}{\hat{A}_n(t) - t} \left\{ 1 - V_{i,n}^{\frac{\hat{A}_n(t)-t}{t}} \right\} \right].$$

The handling of the integral appearing in (12) is slightly more complicated since its computation requires numerical integration as can be seen from the following result also proved in the section "Supporting material" of the online version of the paper.

Proposition 4. *For any $k \in \{1, \ldots, N\}$ and any $t \in (0, 1)$, one has*

$$\int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x \log x} = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_{i,k} - \bar{Z}_k) \log \left(\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t} \right)$$
$$- \frac{1}{\sqrt{n}} \{ \hat{A}_n(t) - tA_n'(t) \} \sum_{i=1}^{n} Z_{i,k} \int_0^1 x^{\hat{A}_n(t)-(1-t)} \left\{ \mathbf{1}(U_{i,n} \leq x^{1-t}) - \frac{\lfloor x^{1-t}(n+1) \rfloor}{n} \right\} \frac{\mathrm{d}x}{x \log x}$$
$$- \frac{1}{\sqrt{n}} \{ \hat{A}_n(t) + (1-t)A_n'(t) \} \sum_{i=1}^{n} Z_{i,k} \int_0^1 x^{\hat{A}_n(t)-t} \left\{ \mathbf{1}(V_{i,n} \leq x^{t}) - \frac{\lfloor x^{t}(n+1) \rfloor}{n} \right\} \frac{\mathrm{d}x}{x \log x},$$

where, for any $y \geq 0$, $\lfloor y \rfloor$ denotes the integer part of y.

4 Extension to LTD copulas

The previous test of exchangeability can actually be extended to a broader class of dependence structures that are not necessarily extreme-value copulas. This broader class consists of copulas that are left-tail decreasing (LTD) in both arguments (see e.g. Nelsen, 2006, Section 5.2.2). From Nelsen (2006, Exercise 5.35), a copula C is LTD in both arguments if and only if, for any $0 < u \le u' \le 1$ and $0 < v \le v' \le 1$,

$$\frac{C(u,v)}{uv} \ge \frac{C(u',v')}{u'v'}.$$

As shown by Garralda-Guillem (2000), extreme-value copulas are LTD in both arguments but so are the most popular bivariate copulas with positive dependence such as the Clayton, Frank, normal, t and Plackett.

4.1 Limiting behavior of A_n^{P} and A_n^{CFG} for LTD copulas

If the second-order partial derivatives of C satisfy some smoothness conditions that shall be explicitly stated below, and if C is LTD in both arguments but is not necessarily an extreme-value copula, we have from the work of Segers (2011) and Genest et al. (2011a) that the nonparametric rank-based estimators A_n^{P} and A_n^{CFG} defined in Section 2 converge uniformly in probability to

$$A_C^{\mathrm{P}}(t) = 1 \Big/ \int_0^1 C(x^{1-t}, x^t) \frac{\mathrm{d}x}{x}, \qquad t \in [0,1],$$

and

$$A_C^{\mathrm{CFG}}(t) = \exp\left[-\gamma + \int_0^1 \left\{ C(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1}) \right\} \frac{\mathrm{d}x}{x \log x} \right], \qquad t \in [0,1],$$

respectively. The functions A_C^{P} and A_C^{CFG} actually turn out to be well-defined for any copula C, and reduce to the Pickands dependence function A when C is an extreme-value copula; see Genest et al. (2011a) for illustrations.

Let $C^{(1,1)}$, $C^{(2,2)}$, and $C^{(1,2)}$ be the second-order partial derivatives of C. The following smoothness conditions were considered by Segers (2011):

(A1) $C^{(1,1)}$ is continuous on $(0,1) \times [0,1]$, and there exists a constant $K_{1,1} > 0$ such that

$$\left| C^{(1,1)}(u,v) \right| \le K_{1,1} \frac{1}{u(1-u)}, \qquad (u,v) \in (0,1) \times [0,1],$$

(A2) $C^{(2,2)}$ is continuous on $[0,1] \times (0,1)$, and there exists a constant $K_{2,2} > 0$ such that

$$\left| C^{(2,2)}(u,v) \right| \le K_{2,2} \frac{1}{v(1-v)}, \qquad (u,v) \in [0,1] \times (0,1),$$

(A3) $C^{(1,2)}$ is continuous on $(0,1)^2$, and there exists a constant $K_{1,2} > 0$ such that

$$C^{(1,2)}(u,v) \leq K_{1,2} \min\left\{\frac{1}{u(1-u)}, \frac{1}{v(1-v)}\right\}, \qquad (u,v) \in (0,1)^2.$$

As shown in Segers (2011, Section 5), these conditions are for instance satisfied for extreme-value copulas whose Pickands dependence function A is twice continuously differentiable on $(0,1)$ with $\sup_{t\in(0,1)}\{t(1-t)A''(t)\} < \infty$, or for the normal copula as long as its dependence parameter is different from -1 and 1. They also appear to hold for many other frequently used bivariate copulas with positive dependence as long as they do not coincide with the upper Fréchet-Hoeffding bound.

From the work of Segers (2011, Section 5) and Proposition 2 of Genest et al. (2011a), we have that, if C satisfies (A1), (A2) and (A3) and is LTD in both arguments, $\sqrt{n}(A_n^{\mathrm{P}} - A_C^{\mathrm{P}}) \rightsquigarrow \mathbb{A}_C^{\mathrm{P}}$ and $\sqrt{n}(A_n^{\mathrm{CFG}} - A_C^{\mathrm{CFG}}) \rightsquigarrow \mathbb{A}_C^{\mathrm{CFG}}$ in $\mathcal{C}([0,1])$, where

$$\mathbb{A}_C^{\mathrm{P}}(t) = -\{A_C^{\mathrm{P}}(t)\}^2 \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}, \qquad t \in [0,1],$$

and

$$\mathbb{A}_C^{\mathrm{CFG}}(t) = A_C^{\mathrm{CFG}}(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \qquad t \in [0,1].$$

It therefore seems sensible to consider an extension of the test presented in the previous section to copulas that are LTD in both arguments.

4.2 Extension of the test

Because LTD dependence structures do not reduce to extreme-value copulas, one cannot base the extended test on the multiplier processes defined in (10) as these involve estimators of the partial derivatives derived under the assumption of extreme-value dependence.

Let us first make the unrealistic assumption that the partial derivatives $C^{(1)}$ and $C^{(2)}$ are known, and, for any $k \in \{1,\ldots,N\}$ and any $(u,v) \in [0,1]^2$, let

$$\bar{\mathbb{C}}_{n,k}(u,v) = \hat{\alpha}_{n,k}(u,v) - C^{(1)}(u,v)\hat{\alpha}_{n,k}(u,1) - C^{(2)}(u,v)\hat{\alpha}_{n,k}(1,v), \qquad (16)$$

be the analogue of $\hat{\mathbb{C}}_{n,k}$ defined in (10), where the domain of $C^{(1)}$ and $C^{(2)}$ is extended to the whole of $[0,1]^2$ as explained in Section 2. Next, for any $k \in \{1,\ldots,N\}$, let $\bar{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\bar{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$ be the analogues of $\hat{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\hat{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$ (defined in (11) and (12), respectively) based on $\bar{\mathbb{C}}_{n,k}$.

The following result, proved in the section "Supporting material" of the online version of the paper, can be regarded as an imperfect extension of Theorem 1 to LTD copulas that do not necessarily belong to the extreme-value class.

Theorem 2. *Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be a random sample from c.d.f. $C\{F(x),G(y)\}$ where C satisfies (A1), (A2) and (A3) and is LTD in both arguments. Then,*

$$\left(\sqrt{n}(A_n^{\mathrm{P}} - A_C^{\mathrm{P}}), \bar{\mathbb{A}}_{n,1}^{\mathrm{P}}, \ldots, \bar{\mathbb{A}}_{n,N}^{\mathrm{P}}\right) \rightsquigarrow \left(\mathbb{A}_C^{\mathrm{P}}, \mathbb{A}_1^{\mathrm{P}}, \ldots, \mathbb{A}_N^{\mathrm{P}}\right)$$

and
$$\left(\sqrt{n}(A_n^{\mathrm{CFG}} - A_C^{\mathrm{CFG}}), \bar{\mathbb{A}}_{n,1}^{\mathrm{CFG}}, \ldots, \bar{\mathbb{A}}_{n,N}^{\mathrm{CFG}}\right) \rightsquigarrow \left(\mathbb{A}_C^{\mathrm{CFG}}, \mathbb{A}_1^{\mathrm{CFG}}, \ldots, \mathbb{A}_N^{\mathrm{CFG}}\right)$$
in $\mathcal{C}([0,1])^{\otimes(N+1)}$, *where* $\mathbb{A}_1^{\mathrm{P}}, \ldots, \mathbb{A}_N^{\mathrm{P}}$ *are independent copies of* $\mathbb{A}_C^{\mathrm{P}}$, *and* $\mathbb{A}_1^{\mathrm{CFG}}, \ldots, \mathbb{A}_N^{\mathrm{CFG}}$ *are independent copies of* $\mathbb{A}_C^{\mathrm{CFG}}$.

As mentioned above, since C is LTD in both arguments but not necessarily an extreme-value copula, the estimators of the partial derivatives defined in the previous section cannot be used anymore. In this case, $C^{(1)}$ and $C^{(2)}$ can be estimated using the estimators proposed in Rémillard and Scaillet (2009) and defined by
$$\tilde{C}_n^{(1)}(u,v) = \frac{\hat{C}_n(u+n^{-1/2}, v) - \hat{C}_n(u-n^{-1/2}, v)}{2n^{-1/2}},$$
and
$$\tilde{C}_n^{(2)}(u,v) = \frac{\hat{C}_n(u, v+n^{-1/2}) - \hat{C}_n(u, v-n^{-1/2})}{2n^{-1/2}}.$$
Now, for any $k \in \{1, \ldots, N\}$, let
$$\tilde{\mathbb{C}}_{n,k}(u,v) = \hat{\alpha}_{n,k}(u,v) - \tilde{C}_n^{(1)}(u,v)\hat{\alpha}_{n,k}(u,1) - \tilde{C}_n^{(2)}(u,v)\hat{\alpha}_{n,k}(1,v), \tag{17}$$
and let $\tilde{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\tilde{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$ be the analogues of $\hat{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\hat{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$, respectively, based on $\tilde{\mathbb{C}}_{n,k}$.

Although an analogue of Theorem 2 remains to be proved for $\tilde{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\tilde{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$, we shall still study the finite sample performance of the test based on S_n when its approximate p-values are computed by $N^{-1}\sum_{k=1}^N \mathbf{1}(\tilde{S}_{n,k} \geq S_n)$, where
$$\tilde{S}_{n,k} = \int_0^{\frac{1}{2}} \left\{\tilde{\mathbb{A}}_{n,k}(t) - \tilde{\mathbb{A}}_{n,k}(1-t)\right\}^2 \mathrm{d}t, \qquad k \in \{1, \ldots, N\},$$
and $\tilde{\mathbb{A}}_{n,k}$ stands for either $\tilde{\mathbb{A}}_{n,k}^{\mathrm{P}}$ or $\tilde{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$.

4.3 Consistency of the extended test

Assume that C is a bivariate non-exchangeable copula satisfying (A1), (A2) and (A3) that is LTD in both arguments. The test based on S_n defined in (15) is then consistent provided there exists $t \in (0, 1/2)$ such that $A_C(t) \neq A_C(1-t)$, where A_C denotes either A_C^{P} or A_C^{CFG}. Indeed,
$$\begin{aligned}\sqrt{n}\{A_n(t) - A_n(1-t)\} &= \sqrt{n}\{A_n(t) - A_C(t)\} \\ &\quad - \sqrt{n}\{A_n(1-t) - A_C(1-t)\} + \sqrt{n}\{A_C(t) - A_C(1-t)\}, \qquad t \in [0,1].\end{aligned}$$
From the continuous mapping theorem and the results stated in the Subsection 4.1, $\sqrt{n}\{A_n(t) - A_C(t)\} - \sqrt{n}\{A_n(1-t) - A_C(1-t)\}$ converges weakly to $\mathbb{A}_C(t) - \mathbb{A}_C(1-t)$ in $\mathcal{C}([0,1])$, while, if there exists $t \in (0,1/2)$ such that $A_C(t) \neq A_C(1-t)$, $\sup_{t\in[0,1/2]} \sqrt{n}|A_C(t) - A_C(1-t)|$ converges to infinity as n tends to infinity, and therefore S_n tends to infinity in probability.

The following proposition, whose proof is left to the reader, involves a simple but strong condition under which the extended test of exchangeability is consistent for LTD copulas.

Proposition 5. *If either* $C(u,v) < C(v,u)$ *whenever* $0 < u < v < 1$ *or* $C(u,v) > C(v,u)$ *whenever* $0 < u < v < 1$*, then* $A_C(t) \neq A_C(1-t)$ *for all* $t \in (0, 1/2)$*.*

The condition stated in the previous proposition is satisfied in many cases for asymmetric copulas defined using the simplest version of Khoudraji's device (Khoudraji, 1995; Genest et al., 1998; Liebscher, 2008). Given an exchangeable copula C_θ, Khoudraji's device can be used to define an asymmetric version of it as

$$aC_{\theta,\lambda,\kappa}(u,v) = u^{1-\lambda} v^{1-\kappa} C_\theta(u^\lambda, v^\kappa), \qquad u, v \in [0,1], \tag{18}$$

for arbitrary choices of $\lambda \neq \kappa$, $\lambda, \kappa \in (0,1]$. It is easy to verify that, if C_θ is an extreme-value copula (resp. LTD in both arguments), then the same is true of $aC_{\theta,\lambda,\kappa}$.

Without loss of generality, assume that $0 < \lambda < \kappa \leq 1$. Then, for most common bivariate exchangeable LTD copulas C_θ such as the Clayton, Frank, normal, t or Plackett with positive dependence, it can be verified that $aC_{\theta,\lambda,\kappa}(u,v) > aC_{\theta,\lambda,\kappa}(v,u)$ whenever $0 < u < v < 1$.

The next proposition, whose proof is immediate, uses a weaker condition under which the extended test of exchangeability is consistent for LTD copulas.

Proposition 6. *If there exists* $t \in (0, 1/2)$ *such that either* $C(x^{1-t}, x^t) < C(x^t, x^{1-t})$ *for all* $x \in (0,1)$*, or* $C(x^{1-t}, x^t) > C(x^t, x^{1-t})$ *for all* $x \in (0,1)$*, then* $A_C(t) \neq A_C(1-t)$*.*

To practically assess whether the test is consistent for a given copula C that is LTD in both arguments, one can plot the functions

$$f(x) = \frac{C(x^{1-t}, x^t) - C(x^t, x^{1-t})}{x}, \quad x \in (0,1),$$

and

$$g(x) = -\frac{C(x^{1-t}, x^t) - C(x^t, x^{1-t})}{x \log(x)}, \quad x \in (0,1),$$

for various values of t in $(0, 1/2)$. Figure 1 shows the graphs of the functions f and g for $t = 1/4$ and asymmetric LTD copulas obtained using the simplest version of Khoudraji's device. The copula C_θ in (18) was successively taken to be the Clayton, Frank, normal, t with 4 degrees of freedom and Plackett, all with a Kendall's tau of 0.5. The shape parameters λ and κ were set to 0.3 and 0.8, respectively. The resulting non-exchangeable copulas will be denoted by aCl, aF, aN, at-4 and aP as we continue.

Figure 1 therefore confirms that the proposed test of exchangeability is consistent for the non-exchangeable copulas aCl, aF, aN, at-4 and aP.

4.4 Implementation of the extended test

To implement the above version of the test, we proceed as in the previous section. The following results, proved in the section "Supporting material" of the online version of the

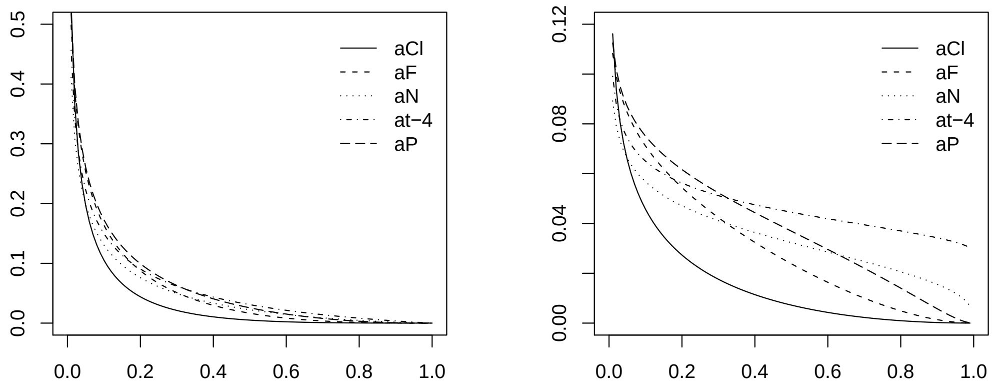

Figure 1: Graphs of the functions f (left) and g (right) for $t = 1/4$ and the asymmetric LTD copulas aCl, aF, aN, at-4 and aP defined in Section 4.

paper, are necessary to compute the integrals appearing in the expressions of $\tilde{\mathbb{A}}^{\mathrm{P}}_{n,k}$ and $\tilde{\mathbb{A}}^{\mathrm{CFG}}_{n,k}$. For every $i \in \{1, \dots, n\}$, let

$$S^+_{i,n} = -\log\left\{(U_{i,n} + n^{-1/2}) \wedge \frac{n}{n+1}\right\}, \quad S^-_{i,n} = -\log\left\{(U_{i,n} - n^{-1/2}) \vee \frac{1}{n+1}\right\},$$

and

$$T^+_{i,n} = -\log\left\{(V_{i,n} + n^{-1/2}) \wedge \frac{n}{n+1}\right\}, \quad T^-_{i,n} = -\log\left\{(V_{i,n} - n^{-1/2}) \vee \frac{1}{n+1}\right\}.$$

Proposition 7. *For any $k \in \{1, \dots, N\}$ and any $t \in (0,1)$, one has*

$$\int_0^1 \tilde{\mathbb{C}}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\left[\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t} \right.$$
$$- \frac{1}{2\sqrt{n}}\sum_{j=1}^n \left\{\frac{S^-_{j,n}}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t} - \frac{S^+_{j,n}}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right.$$
$$\left.\left. + \frac{S_{j,n}}{1-t} \wedge \frac{T^-_{j,n}}{t} \wedge \frac{T_{i,n}}{t} - \frac{S_{j,n}}{1-t} \wedge \frac{T^+_{j,n}}{t} \wedge \frac{T_{i,n}}{t}\right\}\right].$$

Proposition 8. *For any $k \in \{1, \ldots, N\}$ and any $t \in (0,1)$, one has*

$$\int_0^1 \tilde{\mathbb{C}}_{n,k}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x \log x} = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) \left[-\log\left(\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t} \right) \right.$$
$$- \frac{1}{2\sqrt{n}} \sum_{j=1}^n \left\{ -\log\left(\frac{S_{j,n}^-}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t} \right) + \log\left(\frac{S_{j,n}^+}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t} \right) \right.$$
$$\left.\left. - \log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T_{j,n}^-}{t} \wedge \frac{T_{i,n}}{t} \right) + \log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T_{j,n}^+}{t} \wedge \frac{T_{i,n}}{t} \right) \right\} \right].$$

As can be seen from the previous propositions, the computation of the integrals appearing in the expressions of $\tilde{\mathbb{A}}_{n,k}^{\mathrm{P}}$ and $\tilde{\mathbb{A}}_{n,k}^{\mathrm{CFG}}$ is simpler than for the version of the test described in the previous section. Indeed, numerical integration is not necessary anymore, which, as shall be illustrated in Section 6, results in a substantial computational gain.

5 Finite-sample performance

A large scale Monte Carlo experiment was designed to study the finite-sample performance of the proposed test of exchangeability. Four versions of the test were compared. These differ according to whether the generic statistic S_n given in (15) is defined using $A_{n,c}^{\mathrm{CFG}}$ or $A_{n,c}^{\mathrm{P}}$, and according to whether the partial derivatives $C^{(1)}$ and $C^{(2)}$ involved in the computation of the approximate p-values are estimated using the estimators considered in Section 3 or those given in Section 4. As we continue, the resulting procedures will be identified by the statistics $S_{n,c}^{\mathrm{CFG}}$ or $S_{n,c}^{\mathrm{P}}$ when based on the approach for extreme-value copulas described in Section 3, and by $\tilde{S}_{n,c}^{\mathrm{CFG}}$ or $\tilde{S}_{n,c}^{\mathrm{P}}$ when based on the extension to LTD copulas described in Section 4.

The above four versions of the proposed test were also compared with the more general tests of exchangeability studied by Genest et al. (2011b) based on the statistics

$$T_n^{[1]} = \int_{[0,1]^2} n\{\hat{C}_n(u,v) - \hat{C}_n(v,u)\}^2 \mathrm{d}u\mathrm{d}v$$

and

$$T_n^{[2]} = \int_{[0,1]^2} n\{\hat{C}_n(u,v) - \hat{C}_n(v,u)\}^2 \mathrm{d}\hat{C}_n(u,v) = \sum_{i=1}^n \{\hat{C}_n(U_{i,n}, V_{i,n}) - \hat{C}_n(V_{i,n}, U_{i,n})\}^2.$$

Approximate p-values for the above statistics were obtained using a multiplier approach similar to that described in Segers (2011).

To study the level of the tests, data were generated from six exchangeable copulas: the Gumbel-Hougaard (GH), Clayton (Cl), Frank (F), normal (N), t with 4 degrees of freedom (t-4) and Plackett (P). Three levels of dependence were considered corresponding respectively to a Kendall's tau of 0.25, 0.50, and 0.75. Given that the most frequently used bivariate exchangeable extreme-value copulas such as the Gumbel–Hougaard, Galambos,

Table 1: Percentage of rejection of H_0 computed from 1000 random samples of size 50 and 200 generated from exchangeable copulas. Notice that the rejection rates of the tests based on $S_{n,c}^{\mathrm{CFG}}$ and $S_{n,c}^{\mathrm{P}}$ were computed only for data sets generated from the Gumbel-Hougaard copula since these tests are extreme-value copula specific.

n	True	$\tau = 0.25$						$\tau = 0.50$					
		$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	GH	5.2	3.9	3.7	3.7	4.3	3.7	4.7	3.0	3.2	2.1	1.4	3.0
	Cl		4.2		2.2	2.0	3.1		3.3		0.8	1.6	4.0
	F		6.0		4.1	3.1	3.8		3.7		1.2	0.5	2.1
	N		6.0		4.0	2.4	3.5		2.4		1.1	1.4	3.3
	P		4.9		4.3	2.9	3.8		4.8		2.7	2.3	4.2
	t-4		6.2		3.2	2.8	3.7		4.5		1.8	1.7	3.1
200	GH	5.8	5.4	5.4	4.5	3.8	3.9	4.4	2.6	4.3	2.7	2.6	3.1
	Cl		4.3		3.6	3.0	2.8		3.7		1.7	1.8	1.7
	F		4.9		3.7	2.6	3.0		4.8		3.5	1.6	1.6
	N		4.9		5.4	4.4	4.5		2.2		1.5	2.5	2.3
	P		4.5		4.4	2.4	2.2		5.0		3.6	3.0	3.5
	t-4		5.1		6.0	4.3	4.0		4.0		3.6	3.0	2.9

Hüsler–Reiss and Student extreme-value show striking similarities for a given degree of dependence (see Genest et al., 2011a, for a detailed discussion of this matter), only the Gumbel-Hougaard (GH) was used in the study.

The power of the tests was investigated using data sets generated from asymmetric versions of the above mentioned exchangeable copulas constructed using (18). In the experiments, the parameter θ of $aC_{\theta,\lambda,\kappa}$ was set so that the exchangeable copula C_θ involved in its expression has a Kendall's tau of either 0.25, 0.5 or 0.75. The shape parameter λ was allowed to vary in the set $\{0.2, 0.4, 0.6, 0.8\}$, while the parameter κ was kept equal to 1. These values were chosen so that the data generated from the corresponding copulas display various degrees of asymmetry.

The tests based on the statistics $S_{n,c}^{\mathrm{CFG}}$, $S_{n,c}^{\mathrm{P}}$, $\tilde{S}_{n,c}^{\mathrm{CFG}}$, and $\tilde{S}_{n,c}^{\mathrm{P}}$ were carried out using a grid of $m = 100$ uniformly-spaced points on $(0, 1/2)$, while $T_n^{[1]}$ was computed using a grid of $m' = 900$ uniformly-spaced points on $(0, 1)^2$. All the tests were carried out at the 5% significance level and empirical rejection rates were computed from 1000 random samples per scenario. The full simulation results are available in the section "Supporting material" of the online version of the paper. Table 1 partially summarizes the rejection percentages obtained for data sets generated from exchangeable copulas, while Tables 2, 3 and 4 partially report rejection rates when data arise from the asymmetric Gumbel-Hougaard, the asymmetric Clayton and the asymmetric normal, respectively. Notice that the rejection rates of the tests based on $S_{n,c}^{\mathrm{CFG}}$ and $S_{n,c}^{\mathrm{P}}$ were computed only for data sets generated from the Gumbel-Hougaard copula and its asymmetric version since these tests are extreme-value copula specific.

As can be seen from Table 1, for weakly dependent data sets ($\tau = 0.25$), the tests

Table 2: Percentage of rejection of H_0 computed from 1000 random samples of size 50 and 200 generated from the asymmetric Gumbel-Hougaard copula.

n	True	$\theta = 1.33$ $(\tau = 0.25)$						$\theta = 2$ $(\tau = 0.5)$					
		$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aGH}_{\theta,0.8,1}$	5.2	4.6	4.8	4.4	3.1	2.6	9.8	10.2	6.3	5.3	3.7	5.0
	$\mathrm{aGH}_{\theta,0.6,1}$	7.8	6.9	5.0	5.1	4.3	3.7	20.8	21.0	12.0	11.2	8.1	9.5
	$\mathrm{aGH}_{\theta,0.4,1}$	8.5	7.2	4.9	5.3	4.1	4.3	32.0	29.6	16.2	15.4	15.4	15.2
	$\mathrm{aGH}_{\theta,0.2,1}$	8.6	8.2	6.4	7.2	5.1	5.5	27.6	25.6	13.0	13.2	12.0	11.2
200	$\mathrm{aGH}_{\theta,0.8,1}$	8.5	8.4	7.2	6.5	4.7	4.7	34.7	34.7	24.0	21.6	18.0	17.1
	$\mathrm{aGH}_{\theta,0.6,1}$	16.1	16.2	9.4	9.3	8.6	7.8	77.4	78.9	54.3	51.7	50.8	48.5
	$\mathrm{aGH}_{\theta,0.4,1}$	24.5	24.0	14.2	14.0	11.4	11.2	92.8	93.4	69.4	67.7	66.2	65.1
	$\mathrm{aGH}_{\theta,0.2,1}$	21.7	20.5	11.2	10.9	10.5	9.6	82.9	81.8	47.9	46.5	46.8	46.8

Table 3: Percentage of rejection of H_0 computed from 1000 random samples of size 50 and 200 generated from the asymmetric Clayton copula.

n	True	$\theta = 0.67$ $(\tau = 0.25)$				$\theta = 2$ $(\tau = 0.5)$				$\theta = 6$ $(\tau = 0.75)$			
		$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aCl}_{\theta,0.8,1}$	7.0	5.8	4.3	4.8	8.8	7.8	2.2	5.2	17.8	28.5	16.3	31.1
	$\mathrm{aCl}_{\theta,0.6,1}$	6.7	3.8	3.6	4.1	8.4	10.3	5.6	6.9	36.9	50.3	38.7	49.5
	$\mathrm{aCl}_{\theta,0.4,1}$	5.2	4.7	4.0	4.3	11.1	10.6	7.7	8.2	36.4	40.6	33.9	33.8
	$\mathrm{aCl}_{\theta,0.2,1}$	6.0	5.5	4.5	4.8	6.7	8.0	4.6	4.8	14.9	15.9	12.3	12.5
200	$\mathrm{aCl}_{\theta,0.8,1}$	4.6	5.3	3.8	3.7	16.3	39.2	13.5	14.3	71.6	98.6	93.2	95.4
	$\mathrm{aCl}_{\theta,0.6,1}$	7.0	8.1	4.8	4.9	27.7	57.2	28.4	28.4	96.5	100.0	99.8	99.9
	$\mathrm{aCl}_{\theta,0.4,1}$	6.3	7.1	4.7	4.5	23.6	46.0	24.5	25.4	93.3	99.1	97.3	97.2
	$\mathrm{aCl}_{\theta,0.2,1}$	5.8	6.0	4.6	4.7	12.1	16.9	11.4	10.4	54.8	61.8	51.2	50.8

based on $S_{n,c}^{\mathrm{CFG}}$, $S_{n,c}^{\mathrm{P}}$ and $\tilde{S}_{n,c}^{\mathrm{CFG}}$ appear to maintain their nominal level reasonably well, while those based on $\tilde{S}_{n,c}^{\mathrm{P}}$, $T_n^{[1]}$ and $T_n^{[2]}$ are slightly conservative, although the empirical levels improve as n increases. As τ increases, the empirical levels of the tests based on $S_{n,c}^{\mathrm{CFG}}$ and $S_{n,c}^{\mathrm{P}}$ remain reasonably close to 5%. The other four tests, however, are too conservative. As n increases, the agreement with the nominal level appears to improve, although the improvement seems to be very slow, especially for $\tau = 0.75$. Notice that the same phenomenom was observed by Genest et al. (2011b) in their Monte Carlo study. The fact that the tests based on $S_{n,c}^{\mathrm{CFG}}$ and $S_{n,c}^{\mathrm{P}}$ hold their nominal level better than the four other tests might be explained by the slower convergence of the estimators of the partial derivatives used in Section 4 compared to those used in Section 3. Additional simulations indicate that, for $\tau = 0.75$, samples of several thousands of observations are necessary to obtain decent empirical levels for the tests based on $\tilde{S}_{n,c}^{\mathrm{CFG}}$, $\tilde{S}_{n,c}^{\mathrm{P}}$, $T_n^{[1]}$ and $T_n^{[2]}$.

The results reported in Table 2 indicate that, for data arising from the asymmetric

Table 4: Percentage of rejection of H_0 computed from 1000 random samples of size 50 and 200 generated from the asymmetric normal copula.

n	True	$\theta = 0.38$ ($\tau = 0.25$)				$\theta = 0.71$ ($\tau = 0.5$)				$\theta = 0.92$ ($\tau = 0.75$)			
		$\tilde{S}_{n,c}^{\text{CFG}}$	$\tilde{S}_{n,c}^{\text{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\text{CFG}}$	$\tilde{S}_{n,c}^{\text{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\text{CFG}}$	$\tilde{S}_{n,c}^{\text{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\text{aN}_{\theta,0.8,1}$	4.9	3.9	3.3	3.9	6.5	4.6	3.8	5.6	25.4	20.6	12.0	21.2
	$\text{aN}_{\theta,0.6,1}$	5.8	4.9	4.2	4.8	12.2	9.3	6.3	6.8	62.7	47.2	40.4	49.3
	$\text{aN}_{\theta,0.4,1}$	6.5	5.4	4.4	5.1	18.2	13.1	10.4	11.5	69.9	51.4	47.1	48.2
	$\text{aN}_{\theta,0.2,1}$	7.6	6.3	4.5	4.3	11.5	10.8	9.3	8.1	41.3	28.2	23.3	22.4
200	$\text{aN}_{\theta,0.8,1}$	4.4	5.4	4.6	4.4	19.1	18.6	12.1	10.0	93.6	94.8	91.7	88.4
	$\text{aN}_{\theta,0.6,1}$	6.2	5.3	4.2	4.2	45.4	44.2	32.9	28.8	100.0	100.0	99.9	99.9
	$\text{aN}_{\theta,0.4,1}$	7.4	6.1	4.5	4.3	57.0	53.3	39.2	37.0	100.0	99.9	99.8	99.9
	$\text{aN}_{\theta,0.2,1}$	7.1	7.1	4.1	4.0	47.1	35.8	28.7	29.5	98.9	85.8	83.0	83.6

Gumbel-Hougaard copula, there is almost no difference between $S_{n,c}^{\text{CFG}}$ and $\tilde{S}_{n,c}^{\text{CFG}}$, and $S_{n,c}^{\text{P}}$ and $\tilde{S}_{n,c}^{\text{P}}$, respectively. As expected, the empirical powers of all tests increase as θ increases and as the sample size increases. The relationship between the shape parameter λ and the power does not appear to be monotone. The pattern is most visible for $\theta = 2$. The rejection rate increases first as λ decreases, attains its maximum for $\lambda = 0.4$, and then decreases as λ drops down to 0.2. The tests based on $S_{n,c}^{\text{CFG}}$ and $\tilde{S}_{n,c}^{\text{CFG}}$ clearly outperform the other tests, which are comparable in most scenarios. For instance, for sample size 100 and $\theta = 2$ (see the full simulation results in the section "Supporting material" of the online version of the paper), the rejection rates of $S_{n,c}^{\text{CFG}}$ and $\tilde{S}_{n,c}^{\text{CFG}}$ are almost twice that of the other four tests for all four values of λ under consideration.

As can be seen from Table 3, for data sets generated from the asymmetric Clayton copula, the highest rejection rates are obtained for the test based on $\tilde{S}_{n,c}^{\text{P}}$. The same seems true, overall, when data arise from the asymmetric Frank copula, although the difference between the rejection rates of the four tests is much smaller. When data arise from the asymmetric normal (see Table 4), asymmetric t with four degrees of freedom, or asymmetric Plackett copula, the highest rejection rates, overall, are obtained for the test based on $\tilde{S}_{n,c}^{\text{CFG}}$.

6 Illustration

The simulation results described in the previous section show that the proposed tests of exchangeability and their extensions to left-tail decreasing copulas can be substantially more powerful than tests of exchangeability derived directly from the empirical copula.

As an illustration, we applied the tests based on $S_{n,c}^{\text{CFG}}$ and $\tilde{S}_{n,c}^{\text{CFG}}$ to the well-known bivariate data from the Insurance Services Office, Inc., studied in Frees and Valdez (1998). These consist of an indemnity payment and the corresponding allocated loss adjustment expense for 1500 general liability claims randomly chosen from late settlement lags.

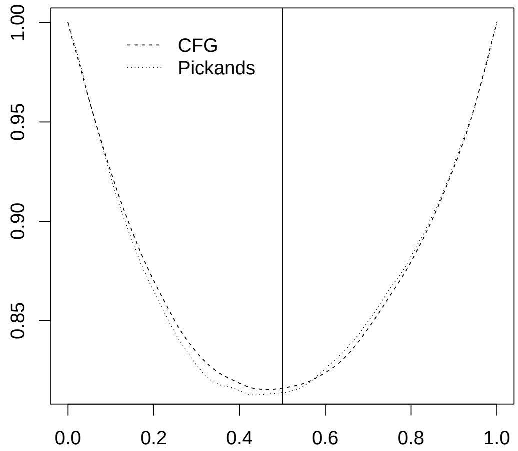

Figure 2: Nonparametric estimates $A_{n,c}^{\mathrm{P}}$ and $A_{n,c}^{\mathrm{CFG}}$ for the Insurance Services Office, Inc data studied in Frees and Valdez (1998).

Among these, 34 claims for which the policy limit was reached were ignored. Many studies, including that of Ben Ghorbal et al. (2009), have concluded that an extreme-value copula is likely to provide an adequate model of the dependence, which we also assumed. Since there are ties present in the data, we used mid-ranks to compute the scaled ranks. Note that a possibly more satisfactory way of dealing with ties was proposed in Kojadinovic and Yan (2010a). The latter approach was not used however, as it turned out to be too computationally expensive in the considered context.

A first step in the modeling of these data consists of plotting the nonparametric estimates $A_{n,c}^{\mathrm{P}}$ and $A_{n,c}^{\mathrm{CFG}}$ of the unknown Pickands dependence function A. The graphs presented in Figure 2 suggest that A might be slightly asymmetric with respect to the axis $t = 1/2$. To test the asymmetry of A more formally, we used the tests based on $S_{n,c}^{\mathrm{CFG}}$ an $\tilde{S}_{n,c}^{\mathrm{CFG}}$ with $N = 10\,000$, and obtained approximate p-values of 12.2% and 12.1%, respectively. As there appears to be only very weak evidence of asymmetry, we opted for an exchangeable extreme-value copula. Because of the lack of diversity in the class of exchangeable extreme-value copulas (see Genest et al., 2011a, for more details), we decided to fit a Gumbel-Hougaard copula by the method-of-moments based on the inversion of Kendall's tau and found a parameter estimate of 1.44 with a standard error of 0.033.

The execution of the test based on $S_{n,c}^{\mathrm{CFG}}$, as implemented in the R package `copula`, took approximately 8 minutes on one 2.2 Ghz processor, while the execution of that based on $\tilde{S}_{n,c}^{\mathrm{CFG}}$ took only 2 minutes. As mentioned in Section 4, this difference is due to the fact that, unlike for the test based on $\tilde{S}_{n,c}^{\mathrm{CFG}}$, numerical integration is necessary to carry out the test based on $S_{n,c}^{\mathrm{CFG}}$

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Supporting material

Additional supporting material may be found in the online version of this article, and consists of the proofs of Propositions 2, 3, 4, 7 and 8., Theorems 1 and 2, and the full simulation results.

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A Proof of Proposition 2

Proof. We split the supremum over $t \in [0,1]$ according to the cases $t \in [0, n^{-1/2})$, $t \in [n^{-1/2}, 1-n^{-1/2}]$, and $t \in (1-n^{-1/2}, 1]$. For $t \in [0, n^{-1/2})$, we have

$$A'_n(t) = \frac{A(2n^{-1/2}) - A(0)}{2n^{-1/2}} + \frac{\mathbb{A}_n(2n^{-1/2}) - \mathbb{A}_n(0)}{2},$$

where $\mathbb{A}_n = \sqrt{n}(A_n - A)$. It follows that

$$\sup_{t\in[0,n^{-1/2})} |A'_n(t) - A'(t)| \leq \sup_{t\in[0,n^{-1/2})} \left| \frac{A(2n^{-1/2}) - A(0)}{2n^{-1/2}} - A'(t) \right| + \frac{1}{2} \left| \mathbb{A}_n(2n^{-1/2}) - \mathbb{A}_n(0) \right|.$$

From the fact that $t \vee (1-t) \leq A(t) \leq 1$ for all $t \in [0,1]$, we have that $A'(t) \in [-1,1]$ for all $t \in (0,1)$, and, from the convexity of A on $[0,1]$, we have that A' is incresing on $(0,1)$. It follows that A' can be extended by continuity at 0 and 1. Since A' exists on $[0,1]$, from the mean value theorem, there exists $t_n \in (0, 2n^{-1/2})$ such that

$$\frac{A(2n^{-1/2}) - A(0)}{2n^{-1/2}} = A'(t_n).$$

It follows that

$$\sup_{t\in[0,n^{-1/2})} \left| \frac{A(2n^{-1/2}) - A(0)}{2n^{-1/2}} - A'(t) \right| = \sup_{t\in[0,n^{-1/2})} |A'(t_n) - A'(t)| \leq \sup_{\substack{x,y\in[0,1] \\ |x-y|\leq 2n^{-1/2}}} |A'(x) - A'(y)|.$$

Since A' is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. Hence, the term on the right converges to zero as n tends to infinity. The fact that $\left|\mathbb{A}_n(2n^{-1/2}) - \mathbb{A}_n(0)\right| =$

$\left|\mathbb{A}_n(2n^{-1/2})\right|$ converges to zero in probability follows from the weak convergence of $\mathbb{A}_n$ in $\mathcal{C}([0,1])$ to the Gaussian process with continuous sample paths $\mathbb{A}$.

Similarly, in the second case,

$$\sup_{t\in[n^{-1/2},1-n^{-1/2}]} |A'_n(t) - A'(t)| \le \sup_{t\in[n^{-1/2},1-n^{-1/2}]} \left| \frac{A(t+n^{-1/2}) - A(t-n^{-1/2})}{2n^{-1/2}} - A'(t) \right| + \frac{1}{2} \sup_{t\in[n^{-1/2},1-n^{-1/2}]} \left|\mathbb{A}_n(t+n^{-1/2}) - \mathbb{A}_n(t-n^{-1/2})\right|.$$

Furthermore,

$$\sup_{t\in[n^{-1/2},1-n^{-1/2}]} \left| \frac{A(t+n^{-1/2}) - A(t-n^{-1/2})}{2n^{-1/2}} - A'(t) \right| = \sup_{t\in[n^{-1/2},1-n^{-1/2}]} |A'(t_n) - A'(t)| \le \sup_{\substack{x,y\in[0,1]\\ |x-y|\le n^{-1/2}}} |A'(x) - A'(y)|,$$

where $t_n \in (t - n^{-1/2}, t + n^{-1/2})$. The term on the right converges to zero as n tends to infinity since A' is uniformly continuous on $[0,1]$. The fact that

$$\sup_{t\in[n^{-1/2},1-n^{-1/2}]} \left|\mathbb{A}_n(t+n^{-1/2}) - \mathbb{A}_n(t-n^{-1/2})\right| \xrightarrow{\Pr} 0$$

follows from the asymptotic equicontinuity of the sequence $\mathbb{A}_n$. The third case is similar. □

B Proof of Theorem 1

Recall that $(U_i, V_i) = (F(X_i), G(Y_i))$ for all $i \in \{1,\dots,n\}$, that C_n is the empirical c.d.f. computed from $(U_1,V_1),\dots,(U_n,V_n)$, and that $\alpha_n = \sqrt{n}(C_n - C)$. Also, let

$$\mathbb{E} = \{(u,v) \in [0,1]^2 : 0 < u \wedge v < 1\} = (0,1]^2 \setminus \{(1,1)\},$$

let $\omega \in (0,1/2)$, let $q_\omega(t) = t^\omega(1-t)^\omega$, $t \in [0,1]$, and let

$$\mathbb{G}_{n,\omega}(u,v) = \begin{cases} \dfrac{\alpha_n(u,v)}{q_\omega(u\wedge v)} & \text{if } (u,v)\in\mathbb{E}, \\ 0 & \text{if } u=0 \text{ or } v=0 \text{ or } (u,v)=(1,1). \end{cases}$$

Genest and Segers (2009, Theorem G.1) showed that the process $\mathbb{G}_{n,\omega}$ converges weakly in $\ell^\infty([0,1]^2)$ to a centered Gaussian process $\mathbb{G}_\omega$ with continuous sample paths such that $\mathbb{G}_\omega(u,v) = 0$ if $u=0$ or $v=0$ or $(u,v)=(1,1)$.

Now, for any $k \in \{1,\dots,N\}$, define

$$\mathbb{G}_{n,\omega,k}(u,v) = \begin{cases} \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) \frac{\mathbf{1}(U_i \le u, V_i \le v)}{q_\omega(u\wedge v)} & \text{if } (u,v)\in\mathbb{E}, \\ 0 & \text{if } u=0 \text{ or } v=0 \text{ or } (u,v)=(1,1). \end{cases}$$

Lemma 1. *We have*

$$(\mathbb{G}_{n,\omega}, \mathbb{G}_{n,\omega,1}, \dots, \mathbb{G}_{n,\omega,N}) \rightsquigarrow (\mathbb{G}_{\omega}, \mathbb{G}_{\omega,1}, \dots, \mathbb{G}_{\omega,N})$$

in $\ell^\infty([0,1]^2)^{\otimes(N+1)}$*, where* $\mathbb{G}_{\omega,1}, \dots, \mathbb{G}_{\omega,N}$ *are independent copies of* $\mathbb{G}_\omega$*.*

Proof. Following Genest and Segers (2009), for any fixed $(u,v) \in \mathbb{E}$, let

$$f_{(u,v),\omega}(s,t) = \frac{\mathbf{1}(s \le u, t \le v) - C(u,v)}{q_\omega(u \wedge v)}, \qquad (s,t) \in \mathbb{E},$$

and let $\mathcal{F}_\omega = \{f_{(u,v),\omega} : (u,v) \in \mathbb{E}\} \cup \{0\}$, where 0 denotes the function vanishing everywhere on $\mathbb{E}$. Furthermore, let P be the probability distribution on $\mathbb{E}$ corresponding to C. Lemma G.2. of Genest and Segers (2009) then states, using the terminology of the theory of empirical processes (see e.g. van der Vaart and Wellner, 2000; Kosorok, 2008), that the collection $\mathcal{F}_\omega$ is P-Donsker, i.e., that there exists a P-Brownian bridge $\mathbb{G}$ such that $\mathbb{G}_n \rightsquigarrow \mathbb{G}$ in $\ell^\infty(\mathcal{F}_\omega)$, where, for each $f \in \mathcal{F}_\omega$,

$$\mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n f - Pf)$$

with

$$\mathbb{P}_n f = \frac{1}{n}\sum_{i=1}^n f(U_i, V_i) \qquad \text{and} \qquad Pf = \int_{[0,1]^2} f(s,t)\mathrm{d}C(s,t).$$

Now, for any $k \in \{1, \dots, N\}$, let

$$\mathbb{G}_{n,k} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\delta_{(U_i,V_i)},$$

where $\delta_{(U_i,V_i)}$ is the probability measure that assigns a mass of 1 to (U_i, V_i). Then, from the functional multiplier central limit theorem (see e.g. Kosorok, 2008, Theorem 10.1), we have that

$$(\mathbb{G}_n, \mathbb{G}_{n,1}, \dots, \mathbb{G}_{n,N}) \rightsquigarrow (\mathbb{G}, \mathbb{G}_1, \dots, \mathbb{G}_N) \tag{19}$$

in $\ell^\infty(\mathcal{F}_\omega)^{\otimes(N+1)}$, where $\mathbb{G}_1, \dots, \mathbb{G}_N$ are independent copies of $\mathbb{G}$. Next, let T be the continuous map from $\ell^\infty(\mathcal{F}_\omega)$ to $\ell^\infty([0,1]^2)$ defined on page 3020 of Genest and Segers (2009) that transforms $\mathbb{G}_n$ into $\mathbb{G}_{n,\omega}$. For any $k \in \{1, \dots, N\}$, if $(u,v) \in \mathbb{E}$, we have

$$\begin{aligned} T(\mathbb{G}_{n,k})(u,v) = \mathbb{G}_{n,k} f_{(u,v),\omega} &= \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\frac{\mathbf{1}(U_i \le u, V_i \le v) - C(u,v)}{q_\omega(u \wedge v)} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\frac{\mathbf{1}(U_i \le u, V_i \le v)}{q_\omega(u \wedge v)} = \mathbb{G}_{n,\omega,k}(u,v), \end{aligned}$$

and, if $u = 0$ or $v = 0$ or $(u,v) = (1,1)$, then $T(\mathbb{G}_{n,k})(u,v) = 0 = \mathbb{G}_{n,\omega,k}(u,v)$. The desired result then follows from (19) and the continuous mapping theorem. □

For any $k \in \{1,\dots,N\}$ and any $u,v \in [0,1]$, define

$$\hat{\mathbb{G}}_{n,\omega,k}(u,v) = \begin{cases} \dfrac{1}{\sqrt{n}}\displaystyle\sum_{i=1}^n (Z_{i,k}-\bar{Z}_k)\frac{\mathbf{1}(U_{i,n}\le u, V_{i,n}\le v)}{q_\omega(u\wedge v)} = \dfrac{\hat{\alpha}_{n,k}(u,v)}{q_\omega(u\wedge v)} & \text{if } (u,v)\in\mathbb{E},\\ 0 & \text{if } (u,v)\in[0,1]^2\setminus\mathbb{E}. \end{cases}$$

Also, let F_n (resp. G_n) be the univariate empirical c.d.f.s computed from $U_1,\dots,U_n$ (resp. $V_1,\dots,V_n$). Their left-continuous generalized inverses will be respectively denoted by F_n^{-1} and G_n^{-1}.

Lemma 2. *We have*

$$\left(\mathbb{G}_{n,\omega},\hat{\mathbb{G}}_{n,\omega,1},\dots,\hat{\mathbb{G}}_{n,\omega,N}\right)\rightsquigarrow(\mathbb{G}_\omega,\mathbb{G}_{\omega,1},\dots,\mathbb{G}_{n,\omega})$$

in $\ell^\infty([0,1]^2)^{\otimes(N+1)}$.

Proof. Proceeding as in Gänssler and Stute (1987, p 52), for any $i\in\{1,\dots,n\}$, we can write

$$\begin{aligned}\mathbf{1}\{U_i\le F_n^{-1}(u),V_i\le G_n^{-1}(v)\}-C(u,v)&=[\mathbf{1}\{U_i\le u,V_i\le v\}-C(u,v)]\\ +\left[C\{F_n^{-1}(u),G_n^{-1}(v)\}-C(u,v)\right]&+\left[\mathbf{1}\{U_i\le F_n^{-1}(u),V_i\le G_n^{-1}(v)\}-C\{F_n^{-1}(u),G_n^{-1}(v)\}\right]\\ &-[\mathbf{1}\{U_i\le u,V_i\le v\}-C\{u,v\}]\,. \end{aligned}\tag{20}$$

Now, for any $k\in\{1,\dots,N\}$, let

$$\tilde{\alpha}_{n,k}(u,v)=\frac{1}{\sqrt{n}}\sum_{i=1}^n(Z_{i,k}-\bar{Z}_k)\mathbf{1}\{U_i\le F_n^{-1}(u),V_i\le G_n^{-1}(v)\},\qquad (u,v)\in[0,1]^2.$$

Then, starting from (20), for any $(u,v)\in\mathbb{E}$, we obtain

$$\begin{aligned}\frac{\tilde{\alpha}_{n,k}(u,v)}{q_\omega(u\wedge v)}=\mathbb{G}_{n,\omega,k}(u,v)+\frac{[C\{F_n^{-1}(u),G_n^{-1}(v)\}-C(u,v)]}{q_\omega(u\wedge v)}\times\frac{1}{\sqrt{n}}\sum_{i=1}^n(Z_{i,k}-\bar{Z}_k)\\ +\left[\mathbb{G}_{n,\omega,k}\{F_n^{-1}(u),G_n^{-1}(v)\}-\mathbb{G}_{n,\omega,k}(u,v)\right].\end{aligned}\tag{21}$$

The second term is equal to 0 while the last term between square brackets converges to zero in probability uniformly in (u,v) due to the asymptotic equicontinuity of the sequence $\mathbb{G}_{n,\omega,k}$. The desired result then follows from Lemma 1 and the fact that the process defined by

$$\begin{cases}\dfrac{\tilde{\alpha}_{n,k}(u,v)}{q_\omega(u\wedge v)} & \text{if } (u,v)\in\mathbb{E},\\ 0 & \text{if } (u,v)\in[0,1]^2\setminus\mathbb{E},\end{cases}$$

and the process $\hat{\mathbb{G}}_{n,\omega,k}$ are asymptotically equivalent. □

Proof of Theorem 1 for the Pickands estimator. Fix $k\in\{1,\dots,N\}$. Using the change of variable $s=-\log x$, we obtain

$$\int_0^1\hat{\mathbb{C}}_{n,k}(x^{1-t},x^t)\frac{\mathrm{d}x}{x}=\int_0^\infty\hat{\mathbb{C}}_{n,k}(e^{-s(1-t)},e^{-st})\mathrm{d}s,\qquad t\in[0,1],$$

which, using (10), can be written as

$$\int_0^\infty \hat{\mathbb{C}}_{n,k}(e^{-s(1-t)}, e^{-st})\mathrm{d}s = \int_0^\infty \hat{\alpha}_{n,k}(e^{-s(1-t)}, e^{-st})\mathrm{d}s \\ - \int_0^\infty \hat{C}_{A_n}^{(1)}(e^{-s(1-t)}, e^{-st})\hat{\alpha}_{n,k}(e^{-s(1-t)}, 1)\mathrm{d}s - \int_0^\infty \hat{C}_{A_n}^{(2)}(e^{-s(1-t)}, e^{-st})\hat{\alpha}_{n,k}(1, e^{-st})\mathrm{d}s.$$

The right-hand side can be rewritten as $\zeta_n(\hat{\mathbb{G}}_{n,\omega,k})(t)$, where, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$,

$$\zeta_n(B)(t) = \int_0^\infty B(e^{-s(1-t)}, e^{-st})K_1(s,t)\mathrm{d}s \\ - \int_0^\infty B(e^{-s(1-t)}, 1)K_{2,n}(s,t)\mathrm{d}s - \int_0^\infty B(1, e^{-st})K_{3,n}(s,t)\mathrm{d}s,$$

where $K_1(s,t) = q_\omega(e^{-s(1-t)} \wedge e^{-st})$, $K_{2,n}(s,t) = q_\omega(e^{-s(1-t)})\hat{C}_{A_n}^{(1)}(e^{-s(1-t)}, e^{-st})$ and $K_{3,n}(s,t) = q_\omega(e^{-st})\hat{C}_{A_n}^{(2)}(e^{-s(1-t)}, e^{-st})$ for all $s \in (0,\infty)$ and $t \in [0,1]$. Now, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$, let

$$\xi(B)(t) = \int_0^\infty B(e^{-s(1-t)}, e^{-st})K_1(s,t)\mathrm{d}s \\ - \int_0^\infty B(e^{-s(1-t)}, 1)K_2(s,t)\mathrm{d}s - \int_0^\infty B(1, e^{-st})K_3(s,t)\mathrm{d}s,$$

where $K_2(s,t) = q_\omega(e^{-s(1-t)})C^{(1)}(e^{-s(1-t)}, e^{-st})$ and $K_3(s,t) = q_\omega(e^{-st})C^{(2)}(e^{-s(1-t)}, e^{-st})$ for all $s \in (0,\infty)$ and $t \in [0,1]$. From the proof of Theorem 3.2 of Genest and Segers (2009), we know that, for any $i \in \{1,2,3\}$, there exists an integrable function $K_i^* : (0,\infty) \to \mathbb{R}$ such that $K_i(s,t) \le K_i^*(s)$ for all $s \in (0,\infty)$ and $t \in [0,1]$. It then follows that the map ξ is continuous with respect to the topologies of uniform convergence on $\ell^\infty([0,1]^2)$ and $\mathcal{C}([0,1])$. From the continuous mapping theorem, we therefore have that $\xi(\hat{\mathbb{G}}_{n,\omega,k})$ converges weakly in $\mathcal{C}([0,1])$ to

$$\xi(\mathbb{G}_{\omega,k})(t) = \int_0^1 \mathbb{C}_k(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}.$$

Next, let $\tilde{K}_{2,n}(s,t) = q_\omega(e^{-s(1-t)})\{A(t) - tA'(t)\}e^{-s\{\hat{A}_n(t)-(1-t)\}}$, and let us now show that, as functions of $t \in [0,1]$,

$$\int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)}, 1)\tilde{K}_{2,n}(s,t)\mathrm{d}s \qquad \text{and} \qquad \int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)}, 1)K_2(s,t)\mathrm{d}s \tag{22}$$

are asymptotically equivalent. The difference between the two processes is

$$\{A(t) - tA'(t)\}\int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)}, 1)q_\omega(e^{-s(1-t)})\left[e^{-s\{\hat{A}_n(t)-(1-t)\}} - e^{-s\{A(t)-(1-t)\}}\right]\mathrm{d}s.$$

Then, for any $t \in [0,1]$,

$$\begin{aligned}
&\int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| q_\omega(e^{-s(1-t)}) \left|e^{-s\{\hat{A}_n(t)-(1-t)\}} - e^{-s\{A(t)-(1-t)\}}\right| \mathrm{d}s \\
&\quad \le \int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| e^{s(1-t)(1-\omega)} \left|e^{-s\hat{A}_n(t)} - e^{-sA(t)}\right| \mathrm{d}s \\
&\quad \le \int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| e^{s(1-t)(1-\omega)} e^{-s\{\hat{A}_n(t)\wedge A(t)\}} s|\hat{A}_n(t) - A(t)|\mathrm{d}s \\
&\quad \le |\hat{A}_n(t) - A(t)| \times \int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| s e^{-s\{\hat{A}_n(t)\wedge A(t)-(1-t)(1-\omega)\}}\mathrm{d}s,
\end{aligned}$$

where the last but one inequality follows from the fact that $|e^{-x} - e^{-y}| \le e^{-(x\wedge y)}|x-y|$ for all $x, y \ge 0$. Given that $A(t) \ge t \vee (1-t)$ and that, by construction, $\hat{A}_n(t) \ge t \vee (1-t)$, one obtains from the proof of Theorem 3.2 of Genest and Segers (2009) that

$$e^{-s\{\hat{A}_n(t)\wedge A(t)-(1-t)(1-\omega)\}} \le e^{-\omega s/2}.$$

As a consequence,

$$\begin{aligned}
&\sup_{t\in[0,1]} \left[\int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| q_\omega(e^{-s(1-t)}) \left|e^{-s\{\hat{A}_n(t)-(1-t)\}} - e^{-s\{A(t)-(1-t)\}}\right| \mathrm{d}s\right] \\
&\qquad \le \sup_{t\in[0,1]} |\hat{A}_n(t) - A(t)| \times \sup_{t\in[0,1]} \left\{\int_0^\infty \left|\hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1)\right| s e^{-s\omega/2}\mathrm{d}s\right\}.
\end{aligned}$$

From the integrability of $s \mapsto s e^{-s\omega/2}$ on $(0,\infty)$, the continuous mapping theorem and the fact that $\sup_{t\in[0,1]} |\hat{A}_n(t) - A(t)| \overset{\Pr}{\to} 0$, the right-side tends to zero in probability, which implies that the two processes defined in (22) are indeed asymptotically equivalent. From the fact that $\sup_{t\in[0,1]} |\hat{A}_n(t) - A(t)| \overset{\Pr}{\to} 0$ and that $\sup_{t\in[0,1]} |A'_n(t) - A'(t)| \overset{\Pr}{\to} 0$, one can then additionally verify that the two processes

$$\int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1) K_{2,n}(s,t)\mathrm{d}s \qquad \text{and} \qquad \int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1) \tilde{K}_{2,n}(s,t)\mathrm{d}s$$

are asymptotically equivalent since their difference is

$$\{\hat{A}_n(t) - tA'_n(t) - A(t) + tA'(t)\} \int_0^\infty \hat{\mathbb{G}}_{n,\omega,k}(e^{-s(1-t)},1) q_\omega(e^{-s(1-t)}) e^{-s\{\hat{A}_n(t)-(1-t)\}}\mathrm{d}s.$$

The same type of reasoning can be used for the process depending on $K_{3,n}$ to finally conclude that $\xi(\hat{\mathbb{G}}_{n,\omega,k})$ and $\zeta_n(\hat{\mathbb{G}}_{n,\omega,k})$ are also asymptotically equivalent.

Now, let $k_n = 2\log(n+1)$ and let $(\xi_n)_{n\ge1}$ be the sequence of deterministic maps from $\ell^\infty([0,1]^2)$ to $\mathcal{C}([0,1])$ defined, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$, by

$$\begin{aligned}
\xi_n(B)(t) = &\int_0^{k_n} B(e^{-s(1-t)}, e^{-st}) K_1(s,t)\mathrm{d}s \\
&\qquad - \int_0^{k_n} B(e^{-s(1-t)},1) K_2(s,t)\mathrm{d}s - \int_0^{k_n} B(1, e^{-st}) K_3(s,t)\mathrm{d}s.
\end{aligned}$$

Given the bounds on the function K_1, K_2 and K_3 mentioned earlier and as discussed in the proof of Theorem 3.2 of Genest and Segers (2009), the sequence of maps $(\xi_n)_{n\geq 1}$ can be used in the framework of the extended continuous mapping theorem (see e.g. van der Vaart, 1998, Theorem 18.11).

Hence, from Lemma 2 and the extended continuous mapping theorem, we have that

$$\left(\xi_n(\mathbb{G}_{n,\omega}), \xi(\hat{\mathbb{G}}_{n,\omega,1}), \ldots, \xi(\hat{\mathbb{G}}_{n,\omega,N})\right) \rightsquigarrow \left(\xi(\mathbb{G}_{\omega}), \xi(\mathbb{G}_{\omega,1}), \ldots, \xi(\mathbb{G}_{\omega,N})\right)$$

in $\mathcal{C}([0,1])^{\otimes(N+1)}$. From the proof of Theorem 3.2 of Genest and Segers (2009), we know that the processes $\xi_n(\mathbb{G}_{n,\omega})(t)$ and

$$\int_0^1 \sqrt{n}\left\{\hat{C}_n(x^{1-t}, x^t) - C(x^{1-t}, x^t)\right\}\frac{\mathrm{d}x}{x} = \sqrt{n}\{1/A_n(t) - 1/A(t)\}$$

are asymptotically equivalent. Given that $\xi(\hat{\mathbb{G}}_{n,\omega,k})$ and $\zeta_n(\hat{\mathbb{G}}_{n,\omega,k})$ are also asymptotically equivalent, we obtain that

$$\left(\sqrt{n}\{1/A_n(t) - 1/A(t)\}, \int_0^\infty \hat{\mathbb{C}}_{n,1}(e^{-s(1-t)}, e^{-st})\mathrm{d}s, \ldots, \int_0^\infty \hat{\mathbb{C}}_{n,N}(e^{-s(1-t)}, e^{-st})\mathrm{d}s\right)$$

converges weakly to

$$\left(\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}, \int_0^1 \mathbb{C}_1(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}, \ldots, \int_0^1 \mathbb{C}_N(x^{1-t}, x^t)\frac{\mathrm{d}x}{x}\right)$$

in $\mathcal{C}([0,1])^{\otimes(N+1)}$. The result stated in (13) finally follows from the the functional version of Slutsky's theorem, the continuous mapping theorem and the fact that $\sup_{t\in[0,1]}|A_n(t) - A(t)| \overset{\Pr}{\to} 0$. □

Proof of Theorem 1 for the CFG estimator. The proof is very similar to that for the Pickands estimator and uses the same notation. Fix $k \in \{1, \ldots, N\}$. Then, using the change of variable $s = -\log x$,

$$\int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} = -\int_0^\infty \hat{\mathbb{C}}_{n,k}(e^{-s(1-t)}, e^{-st})\frac{\mathrm{d}s}{s} = \vartheta_n(\hat{\mathbb{G}}_{n,\omega,k})(t), \qquad t \in [0,1],$$

where, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$,

$$\vartheta_n(B)(t) = -\int_0^\infty B(e^{-s(1-t)}, e^{-st})K_1(s,t)\frac{\mathrm{d}s}{s} + \int_0^\infty B(e^{-s(1-t)}, 1)K_{2,n}(s,t)\frac{\mathrm{d}s}{s} + \int_0^\infty B(1, e^{-st})K_{3,n}(s,t)\frac{\mathrm{d}s}{s}.$$

Proceeding as for the Pickands estimator, it can verified that $\vartheta_n(\hat{\mathbb{G}}_{n,\omega,k})$ is asymptotically equivalent to $\vartheta(\hat{\mathbb{G}}_{n,\omega,k})$, where, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$,

$$\vartheta(B)(t) = -\int_0^\infty B(e^{-s(1-t)}, e^{-st})K_1(s,t)\frac{\mathrm{d}s}{s} + \int_0^\infty B(e^{-s(1-t)}, 1)K_2(s,t)\frac{\mathrm{d}s}{s} + \int_0^\infty B(1, e^{-st})K_3(s,t)\frac{\mathrm{d}s}{s}.$$

From the proof of Theorem 3.2 of Genest and Segers (2009), we have that the map ϑ is continuous with respect to the topologies of uniform convergence on $\ell^\infty([0,1]^2)$ and $\mathcal{C}([0,1])$. Indeed, for $s \in [1,\infty)$, the same upper bounds K_1^*, K_2^*, K_3^* as for the Pickands estimator can be used for the functions K_1, K_2, K_3 in the previous expression. The integrability on $(0,1)$ follows from the additional bound $|1 - e^{-s(1-t)} \wedge e^{-st}|^\omega \le s^\omega$.

Now, let $(\phi_n)_{n\ge 1}$ be the sequence of deterministic maps from $\ell^\infty([0,1]^2)$ to $\mathcal{C}([0,1])$ defined, for any $B \in \ell^\infty([0,1]^2)$ and any $t \in [0,1]$, by

$$\phi_n(B)(t) = -\int_{l_n}^{k_n} B(e^{-s(1-t)}, e^{-st}) K_1(s,t) \frac{\mathrm{d}s}{s} + \int_{l_n}^{k_n} B(e^{-s(1-t)}, 1) K_2(s,t) \frac{\mathrm{d}s}{s} + \int_{l_n}^{k_n} B(1, e^{-st}) K_3(s,t) \frac{\mathrm{d}s}{s},$$

where $l_n = 1/(n+1)$ and $k_n = 2\log(n+1)$. From the proof of Theorem 3.2 of Genest and Segers (2009), we have that

$$\sqrt{n}\{\log A_n(t) - \log A(t)\} = \int_0^1 \sqrt{n}\{C_n(x^{1-t}, x^t) - C(x^{1-t}, x^t)\} \frac{\mathrm{d}x}{x \log x}, \qquad t \in [0,1],$$

is asymptotically equivalent to the process $\phi_n(\mathbb{G}_{n,\omega})(t)$, $t \in [0,1]$. From Lemma 2 and the extended continuous mapping theorem, we then have that

$$\Big(\phi_n(\mathbb{G}_{n,\omega}), \vartheta(\hat{\mathbb{G}}_{n,\omega,1}), \dots, \vartheta(\hat{\mathbb{G}}_{n,\omega,N})\Big) \rightsquigarrow \Big(\vartheta(\mathbb{G}_\omega), \vartheta(\mathbb{G}_{\omega,1}), \dots, \vartheta(\mathbb{G}_{\omega,N})\Big)$$

in $\mathcal{C}([0,1])^{\otimes(N+1)}$, which implies that

$$\Big(\sqrt{n}(\log A_n - \log A), \vartheta_n(\hat{\mathbb{G}}_{n,\omega,1}), \dots, \vartheta_n(\hat{\mathbb{G}}_{n,\omega,N})\Big) \rightsquigarrow \Big(\vartheta(\mathbb{G}_\omega), \vartheta(\mathbb{G}_{\omega,1}), \dots, \vartheta(\mathbb{G}_{\omega,N})\Big)$$

in $\mathcal{C}([0,1])^{\otimes(N+1)}$. The result stated in (14) finally follows from the the functional version of Slutsky's theorem, the continuous mapping theorem and the fact that $\sup_{t\in[0,1]} |A_n(t) - A(t)| \overset{\Pr}{\to} 0$. □

C Proof of Theorem 2

Proof. The proof is very similar to that of Theorem 1, and relies on the fact that, when C is LTD in both arguments, $K_i(s,t) \le e^{-\omega s/2}$ for all $s \in (0,\infty)$, $t \in [0,1]$ and $i = 2,3$ (Genest et al., 2011a, Appendix C) . For the CFG estimator, one additionally has that $K_i(s,t) \le s^\omega$ for all $s \in (0,1)$, $t \in [0,1]$ and $i = 2,3$. □

D Proofs of Propositions 3, 4, 7 and 8

Proof of Proposition 3. Starting from (10), one obtains

$$\int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\int_0^1 \Big\{\mathbf{1}(U_{i,n} \le x^{1-t})\mathbf{1}(V_{i,n} \le x^t) \\ -\hat{C}_{A_n}^{(1)}(x^{1-t}, x^t)\mathbf{1}(U_{i,n} \le x^{1-t}) - \hat{C}_{A_n}^{(2)}(x^{1-t}, x^t)\mathbf{1}(V_{i,n} \le x^t)\Big\}\frac{\mathrm{d}x}{x}.$$

The result then partly follows from the fact that

$$\int_0^1 \mathbf{1}(U_{i,n} \le x^{1-t})\mathbf{1}(V_{i,n} \le x^t)\frac{\mathrm{d}x}{x} = \int_0^1 \mathbf{1}(U_{i,n}^{\frac{1}{1-t}} \le x)\mathbf{1}(V_{i,n}^{\frac{1}{t}} \le x)\frac{\mathrm{d}x}{x} = \frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}.$$

The desired result is finally obtained from the expressions of $\hat{C}_{A_n}^{(1)}(x^{1-t}, x^t)$ and $\hat{C}_{A_n}^{(2)}(x^{1-t}, x^t)$ given in (7) and (8). □

Proof of Proposition 4. Starting from (10), one can write

$$\int_0^1 \hat{\mathbb{C}}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} = \int_0^1 \hat{\alpha}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} \\ - \int_0^1 \hat{C}_{A_n}^{(1)}(x^{1-t}, x^t)\hat{\alpha}_{n,k}(x^{1-t}, 1)\frac{\mathrm{d}x}{x\log x} - \int_0^1 \hat{C}_{A_n}^{(2)}(x^{1-t}, x^t)\hat{\alpha}_{n,k}(1, x^t)\frac{\mathrm{d}x}{x\log x}.$$

Then, from the definition of $\hat{\alpha}_{n,k}$ given in (9),

$$\int_0^1 \hat{\alpha}_{n,k}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\int_0^1 \mathbf{1}(U_{i,n} \le x^{1-t})\mathbf{1}(V_{i,n} \le x^t)\frac{\mathrm{d}x}{x\log x}.$$

Using the fact that, for every $i \in \{1, \dots, n\}$,

$$\mathbf{1}(U_{i,n} \le x^{1-t})\mathbf{1}(V_{i,n} \le x^t) = \mathbf{1}(U_{i,n}^{\frac{1}{1-t}} \vee V_{i,n}^{\frac{1}{t}} \le x) = \mathbf{1}\left\{-\log(x) \le \frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}\right\},$$

and that $\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) = 0$, one obtains

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\int_0^1 \mathbf{1}(U_{i,n} \le x^{1-t})\mathbf{1}(V_{i,n} \le x^t)\frac{\mathrm{d}x}{x\log x} \\ = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k} - \bar{Z}_k)\int_0^1 \left[\mathbf{1}\left\{-\log(x) \le \frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}\right\} - \mathbf{1}(x \ge e^{-1})\right]\frac{\mathrm{d}x}{x\log x}.$$

Now, as shown in Genest and Segers (2009, Appendix A),

$$-\log(z) = \int_0^1 [\mathbf{1}\{-\log(x) \le z\} - \mathbf{1}(x \ge e^{-1})]\frac{\mathrm{d}x}{x\log x}, \qquad z \in (0, 1). \tag{23}$$

It follows that

$$\int_0^1 \hat{\alpha}_{n,k}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x \log x} = -\frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) \log \left(\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t} \right).$$

For the second integral, starting from (9), one obtains

$$\begin{aligned}
&\int_0^1 \hat{C}^{(1)}_{A_n}(x^{1-t}, x^t) \hat{\alpha}_{n,k}(x^{1-t}, 1) \frac{\mathrm{d}x}{x \log x} \\
&\quad = \frac{1}{\sqrt{n}} \{\hat{A}_n(t) - tA_n'(t)\} \sum_{i=1}^n Z_{i,k} \int_0^1 x^{\hat{A}_n(t)-(1-t)} \left\{ \mathbf{1}(U_{i,n} \le x^{1-t}) - \hat{C}_n(x^{1-t}, 1) \right\} \frac{\mathrm{d}x}{x \log x} \\
&\quad = \frac{1}{\sqrt{n}} \{\hat{A}_n(t) - tA_n'(t)\} \sum_{i=1}^n Z_{i,k} \int_0^1 x^{\hat{A}_n(t)-(1-t)} \left\{ \mathbf{1}(U_{i,n} \le x^{1-t}) - \lfloor x^{1-t}(n+1) \rfloor / n \right\} \frac{\mathrm{d}x}{x \log x}.
\end{aligned}$$

The expression of the third integral is obtained similarly. □

Proof of Proposition 7. We start from (17). The result partly follows from the proof of Proposition 3, where it is shown that

$$\int_0^1 \hat{\alpha}_{n,k}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) \frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}.$$

It is furthermore easy to verify from (9) that $\hat{\alpha}_{n,k}(x^{1-t}, 1) = 0$ if $x^{1-t} \ge n/(n+1)$. Hence,

$$\begin{aligned}
\int_0^1 \tilde{C}^{(1)}_n(x^{1-t}, x^t) \hat{\alpha}_{n,k}(x^{1-t}, 1) \frac{\mathrm{d}x}{x} &= \int_0^1 \tilde{C}^{(1)}_n(x^{1-t}, x^t) \hat{\alpha}_{n,k}(x^{1-t}, 1) \mathbf{1}\left(x^{1-t} \le \frac{n}{n+1} \right) \frac{\mathrm{d}x}{x} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_{i,k} - \bar{Z}_k) \int_0^1 \tilde{C}^{(1)}_n(x^{1-t}, x^t) \mathbf{1}\left(U_{i,n} \le x^{1-t} \le \frac{n}{n+1} \right) \frac{\mathrm{d}x}{x}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^1 \tilde{C}^{(1)}_n(x^{1-t}, x^t) \mathbf{1}\left(U_{i,n} \le x^{1-t} \le \frac{n}{n+1} \right) \frac{\mathrm{d}x}{x} \\
&= \frac{1}{2n^{-1/2}} \int_0^1 \left\{ \hat{C}_n(x^{1-t} + n^{-1/2}, x^t) - \hat{C}_n(x^{1-t} - n^{-1/2}, x^t) \right\} \mathbf{1}\left(U_{i,n} \le x^{1-t} \le \frac{n}{n+1} \right) \frac{\mathrm{d}x}{x}.
\end{aligned}$$

We then have

$$\begin{aligned}
&\int_0^1 \hat{C}_n(x^{1-t}+n^{-1/2},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le \frac{n}{n+1}\right)\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{n}\sum_{j=1}^n\int_0^1 \mathbf{1}(U_{j,n}\le x^{1-t}+n^{-1/2})\mathbf{1}(V_{j,n}\le x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le \frac{n}{n+1}\right)\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{n}\sum_{j=1}^n\int_0^1 \mathbf{1}\left[\left\{(U_{j,n}-n^{-1/2})\vee\frac{1}{n+1}\right\}^{\frac{1}{1-t}}\vee U_{i,n}^{\frac{1}{1-t}}\vee V_{j,n}^{\frac{1}{t}}\le x\le\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\right]\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{1-t}\log\left(\frac{n}{n+1}\right)-\frac{1}{n}\sum_{j=1}^n\log\left[\left\{(U_{j,n}-n^{-1/2})\vee\frac{1}{n+1}\right\}^{\frac{1}{1-t}}\vee U_{i,n}^{\frac{1}{1-t}}\vee V_{j,n}^{\frac{1}{t}}\right]\\
&\quad=\frac{1}{1-t}\log\left(\frac{n}{n+1}\right)+\frac{1}{n}\sum_{j=1}^n\frac{S_{j,n}^-}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \hat{C}_n(x^{1-t}-n^{-1/2},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le \frac{n}{n+1}\right)\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{n}\sum_{j=1}^n\int_0^1 \mathbf{1}(U_{j,n}\le x^{1-t}-n^{-1/2})\mathbf{1}(V_{j,n}\le x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le \frac{n}{n+1}\right)\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{n}\sum_{j=1}^n\int_0^1 \mathbf{1}\left[\left\{(U_{j,n}+n^{-1/2})\wedge\frac{n}{n+1}\right\}\vee U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right]\mathbf{1}(V_{j,n}\le x^t)\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{n}\sum_{j=1}^n\int_0^1 \mathbf{1}\left[\left\{(U_{j,n}+n^{-1/2})\wedge\frac{n}{n+1}\right\}^{\frac{1}{1-t}}\vee U_{i,n}^{\frac{1}{1-t}}\vee V_{j,n}^{\frac{1}{t}}\le x\le\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\right]\frac{\mathrm{d}x}{x}\\
&\quad=\frac{1}{1-t}\log\left(\frac{n}{n+1}\right)-\frac{1}{n}\sum_{j=1}^n\log\left[\left\{(U_{j,n}+n^{-1/2})\wedge\frac{n}{n+1}\right\}^{\frac{1}{t}}\vee U_{i,n}^{\frac{1}{1-t}}\vee V_{j,n}^{\frac{1}{t}}\right]\\
&\quad=\frac{1}{1-t}\log\left(\frac{n}{n+1}\right)+\frac{1}{n}\sum_{j=1}^n\frac{S_{j,n}^+}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}.
\end{aligned}$$

It follows that

$$\int_0^1 \tilde{C}_n^{(1)}(x^{1-t},x^t)\mathbf{1}(U_{i,n}\le x^{1-t})\frac{\mathrm{d}x}{x}=\frac{1}{2\sqrt{n}}\sum_{j=1}^n\left\{\frac{S_{j,n}^-}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}-\frac{S_{j,n}^+}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}\right\}.$$

Similarly, one obtains that

$$\int_0^1 \tilde{C}_n^{(2)}(x^{1-t},x^t)\mathbf{1}(V_{i,n}\le x^{t})\frac{\mathrm{d}x}{x}=\frac{1}{2\sqrt{n}}\sum_{j=1}^n\left\{\frac{S_{j,n}}{1-t}\wedge\frac{T_{j,n}^-}{t}\wedge\frac{T_{i,n}}{t}-\frac{S_{j,n}}{1-t}\wedge\frac{T_{j,n}^+}{t}\wedge\frac{T_{i,n}}{t}\right\}.$$

□

Proof of Proposition 8. We start again from (17). It has already been shown that

$$\int_0^1 \hat{\alpha}_{n,k}(x^{1-t},x^t)\frac{\mathrm{d}x}{x\log x} = -\frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k}-\bar{Z}_k)\log\left(\frac{S_{i,n}}{1-t}\wedge\frac{T_{i,n}}{t}\right).$$

Proceeding as in the proof of Proposition 7, one has

$$\begin{aligned}
&\int_0^1 \tilde{C}_n^{(1)}(x^{1-t},x^t)\hat{\alpha}_{n,k}(x^{1-t},1)\frac{\mathrm{d}x}{x\log x}\\
&\qquad = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z_{i,k}-\bar{Z}_k)\int_0^1 \tilde{C}_n^{(1)}(x^{1-t},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)\frac{\mathrm{d}x}{x\log x},
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1 \tilde{C}_n^{(1)}(x^{1-t},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)\frac{\mathrm{d}x}{x\log x}\\
=&\frac{1}{2n^{-1/2}}\int_0^1\Bigg[\hat{C}_n(x^{1-t}+n^{-1/2},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)-\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\ge x\ge e^{-1}\right\}\\
&-\hat{C}_n(x^{1-t}-n^{-1/2},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)+\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\ge x\ge e^{-1}\right\}\Bigg]\frac{\mathrm{d}x}{x\log x}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^1\left[\hat{C}_n(x^{1-t}+n^{-1/2},x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)-\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\ge x\ge e^{-1}\right\}\right]\frac{\mathrm{d}x}{x\log x}\\
=&\frac{1}{n}\sum_{j=1}^n\int_0^1\Bigg[\mathbf{1}(U_{j,n}\le x^{1-t}+n^{-1/2})\mathbf{1}(V_{j,n}\le x^t)\mathbf{1}\left(U_{i,n}\le x^{1-t}\le\frac{n}{n+1}\right)\\
&-\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\ge x\ge e^{-1}\right\}\Bigg]\frac{\mathrm{d}x}{x\log x}\\
=&\frac{1}{n}\sum_{j=1}^n\int_0^1\Bigg(\mathbf{1}\left[\left\{(U_{j,n}-n^{-1/2})\vee\frac{1}{n+1}\right\}^{\frac{1}{1-t}}\vee U_{i,n}^{\frac{1}{1-t}}\vee V_{j,n}^{\frac{1}{t}}\le x\le\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\right]\\
&-\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\ge x\ge e^{-1}\right\}\Bigg)\frac{\mathrm{d}x}{x\log x}\\
=&\frac{1}{n}\sum_{j=1}^n\int_0^1\left[1\left\{-\log(x)\le\frac{S_{j,n}^-}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}\right\}-\mathbf{1}(x\ge e^{-1})\right]\frac{\mathrm{d}x}{x\log x}\\
=&-\frac{1}{n}\sum_{j=1}^n\log\left(\frac{S_{j,n}^-}{1-t}\wedge\frac{T_{j,n}}{t}\wedge\frac{S_{i,n}}{1-t}\right),
\end{aligned}$$

where the last equality follows from (23). Similarly,

$$
\begin{aligned}
&\int_0^1 \left[\hat{C}_n(x^{1-t} - n^{-1/2}, x^t)\mathbf{1}\left(U_{i,n} \le x^{1-t} \le \frac{n}{n+1}\right) - \mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}} \ge x \ge e^{-1}\right\}\right] \frac{\mathrm{d}x}{x\log x} \\
=& \frac{1}{n}\sum_{j=1}^n \int_0^1 \left[\mathbf{1}(U_{j,n} \le x^{1-t} - n^{-1/2})\mathbf{1}(V_{j,n} \le x^t)\mathbf{1}\left(U_{i,n} \le x^{1-t} \le \frac{n}{n+1}\right)\right. \\
&\left. -\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}} \ge x \ge e^{-1}\right\}\right] \frac{\mathrm{d}x}{x\log x} \\
=& \frac{1}{n}\sum_{j=1}^n \int_0^1 \left(\mathbf{1}\left[\left\{(U_{j,n} + n^{-1/2}) \wedge \frac{n}{n+1}\right\}^{\frac{1}{1-t}} \vee U_{i,n}^{\frac{1}{1-t}} \vee V_{j,n}^{\frac{1}{t}} \le x \le \left(\frac{n}{n+1}\right)^{\frac{1}{1-t}}\right]\right. \\
&\left. -\mathbf{1}\left\{\left(\frac{n}{n+1}\right)^{\frac{1}{1-t}} \ge x \ge e^{-1}\right\}\right) \frac{\mathrm{d}x}{x\log x} \\
=& \frac{1}{n}\sum_{j=1}^n \int_0^1 \left[1\left\{-\log(x) \le \frac{S_{j,n}^+}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right\} - \mathbf{1}(x \ge e^{-1})\right] \frac{\mathrm{d}x}{x\log x} \\
=& -\frac{1}{n}\sum_{j=1}^n \log\left(\frac{S_{j,n}^+}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right),
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^1 \tilde{C}_n^{(1)}(x^{1-t}, x^t)\mathbf{1}(U_{i,n} \le x^{1-t})\frac{\mathrm{d}x}{x\log x} \\
&\qquad = \frac{1}{2\sqrt{n}}\sum_{j=1}^n \left\{-\log\left(\frac{S_{j,n}^-}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right) + \log\left(\frac{S_{j,n}^+}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right)\right\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1 \tilde{C}_n^{(2)}(x^{1-t}, x^t)\mathbf{1}(V_{i,n} \le x^t)\frac{\mathrm{d}x}{x\log x} \\
&\qquad = \frac{1}{2\sqrt{n}}\sum_{j=1}^n \left\{-\log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T_{j,n}^-}{t} \wedge \frac{S_{i,n}}{1-t}\right) + \log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T_{j,n}^+}{t} \wedge \frac{S_{i,n}}{1-t}\right)\right\}.
\end{aligned}
$$

□

E Full simulation results

Table 5: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from exchangeable copulas. Notice that the rejection rates of the tests based on $S_{n,c}^{\mathrm{CFG}}$ and $S_{n,c}^{\mathrm{P}}$ were computed only for data sets generated from the Gumbel-Hougaard copula since these tests are extreme-value copula specific.

n	True	$\tau = 0.25$						$\tau = 0.50$						$\tau = 0.75$					
		$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	GH	5.2	3.9	3.7	3.7	4.3	3.7	4.7	3.0	3.2	2.1	1.4	3.0	3.6	0.6	2.5	0.3	0.1	2.3
	Cl		4.2		2.2	2.0	3.1		3.3		0.8	1.6	4.0		1.8		0.6	0.5	2.8
	F		6.0		4.1	3.1	3.8		3.7		1.2	0.5	2.1		0.7		0.3	0.5	2.1
	N		6.0		4.0	2.4	3.5		2.4		1.1	1.4	3.3		0.3		0.0	0.3	2.2
	P		4.9		4.3	2.9	3.8		4.8		2.7	2.3	4.2		2.3		1.2	0.6	5.4
	t-4		6.2		3.2	2.8	3.7		4.5		1.8	1.7	3.1		1.1		0.6	0.2	2.9
100	GH	4.5	3.8	3.6	2.8	2.5	2.0	3.8	2.7	3.6	2.2	2.0	2.7	5.3	1.1	5.2	0.5	0.5	2.5
	Cl		4.9		2.7	5.0	4.2		5.0		2.0	2.1	2.9		1.7		0.2	0.3	1.7
	F		5.0		4.0	3.2	2.5		4.1		3.3	1.5	1.5		0.9		0.2	0.2	2.0
	N		2.9		3.1	2.3	2.5		2.0		0.6	0.7	0.9		0.5		0.2	0.3	2.2
	P		5.4		5.1	2.8	3.2		4.5		2.5	2.1	2.5		2.9		1.7	0.7	4.7
	t-4		4.7		4.5	2.9	2.9		4.7		2.6	2.3	2.3		1.3		0.3	0.1	1.5
200	GH	5.8	5.4	5.4	4.5	3.8	3.9	4.4	2.6	4.3	2.7	2.6	3.1	3.4	0.6	3.7	0.6	0.6	1.2
	Cl		4.3		3.6	3.0	2.8		3.7		1.7	1.8	1.7		3.1		1.2	1.2	2.0
	F		4.9		3.7	2.6	3.0		4.8		3.5	1.6	1.6		1.8		0.6	0.3	2.1
	N		4.9		5.4	4.4	4.5		2.2		1.5	2.5	2.3		0.6		0.2	0.8	1.0
	P		4.5		4.4	2.4	2.2		5.0		3.6	3.0	3.5		3.7		1.3	1.7	3.8
	t-4		5.1		6.0	4.3	4.0		4.0		3.6	3.0	2.9		1.8		1.0	0.6	1.4
400	GH	4.8	4.2	5.9	5.4	4.2	3.5	4.3	3.4	5.3	3.9	2.6	2.2	3.4	1.6	5.6	1.3	1.4	1.9
	Cl		4.4		3.2	3.2	2.6		4.4		2.6	3.2	2.6		4.0		1.6	1.3	1.4
	F		5.3		4.4	4.1	3.4		4.8		3.5	2.6	2.4		2.0		0.8	0.7	0.7
	N		3.5		2.4	2.5	2.3		4.5		2.0	2.2	1.6		0.3		0.2	0.8	0.1
	P		5.9		5.8	4.9	4.7		5.2		4.2	3.7	3.4		4.9		2.2	2.8	2.7
	t-4		5.6		4.6	2.9	2.8		4.2		2.8	1.6	1.8		2.0		1.1	1.3	1.2

Table 6: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric Gumbel-Hougaard copula.

n	True	$\theta = 1.33$						$\theta = 2$						$\theta = 4$					
		$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$S_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$S_{n,c}^{\mathrm{P}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aGH}_{\theta,0.8,1}$	5.2	4.6	4.8	4.4	3.1	2.6	9.8	10.2	6.3	5.3	3.7	5.0	33.3	35.8	21.9	19.5	16.2	28.3
	$\mathrm{aGH}_{\theta,0.6,1}$	7.8	6.9	5.0	5.1	4.3	3.7	20.8	21.0	12.0	11.2	8.1	9.5	77.9	78.6	56.8	52.1	51.1	59.0
	$\mathrm{aGH}_{\theta,0.4,1}$	8.5	7.2	4.9	5.3	4.1	4.3	32.0	29.6	16.2	15.4	15.4	15.2	86.1	83.9	60.6	55.9	56.4	56.8
	$\mathrm{aGH}_{\theta,0.2,1}$	8.6	8.2	6.4	7.2	5.1	5.5	27.6	25.6	13.0	13.2	12.0	11.2	56.7	52.1	29.7	29.7	26.3	25.6
100	$\mathrm{aGH}_{\theta,0.8,1}$	6.4	6.5	5.4	5.1	4.6	4.8	17.6	18.6	10.7	9.7	8.4	7.9	69.7	75.4	58.2	52.6	51.4	57.3
	$\mathrm{aGH}_{\theta,0.6,1}$	8.8	8.1	4.3	3.8	4.0	3.8	46.2	47.5	27.0	25.2	23.4	22.5	98.6	98.5	93.3	90.1	92.4	93.0
	$\mathrm{aGH}_{\theta,0.4,1}$	13.4	13.1	8.2	8.1	6.4	6.9	64.3	62.6	37.7	36.4	33.7	30.5	99.6	99.7	93.7	91.3	93.1	92.1
	$\mathrm{aGH}_{\theta,0.2,1}$	14.3	13.3	8.0	8.1	6.1	5.9	53.5	51.3	28.2	28.3	26.1	24.2	89.2	87.7	59.3	57.8	55.1	56.2
200	$\mathrm{aGH}_{\theta,0.8,1}$	8.5	8.4	7.2	6.5	4.7	4.7	34.7	34.7	24.0	21.6	18.0	17.1	94.5	96.1	91.4	89.0	88.7	89.6
	$\mathrm{aGH}_{\theta,0.6,1}$	16.1	16.2	9.4	9.3	8.6	7.8	77.4	78.9	54.3	51.7	50.8	48.5	99.9	100.0	99.9	99.8	99.8	99.8
	$\mathrm{aGH}_{\theta,0.4,1}$	24.5	24.0	14.2	14.0	11.4	11.2	92.8	93.4	69.4	67.7	66.2	65.1	100.0	100.0	100.0	100.0	100.0	99.9
	$\mathrm{aGH}_{\theta,0.2,1}$	21.7	20.5	11.2	10.9	10.5	9.6	82.9	81.8	47.9	46.5	46.8	46.8	99.8	99.7	91.9	90.2	92.1	92.9
400	$\mathrm{aGH}_{\theta,0.8,1}$	10.7	10.5	6.4	6.4	6.5	5.8	62.5	63.3	43.4	39.6	36.5	33.6	100.0	100.0	100.0	99.9	99.8	99.8
	$\mathrm{aGH}_{\theta,0.6,1}$	21.9	22.0	12.6	12.6	11.0	10.2	98.0	98.2	85.0	83.4	85.1	83.7	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{aGH}_{\theta,0.4,1}$	39.2	38.9	22.0	21.2	19.6	18.1	99.9	100.0	93.0	92.5	93.7	93.5	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{aGH}_{\theta,0.2,1}$	39.8	39.4	21.6	21.4	18.4	17.5	98.9	98.7	83.3	81.9	83.8	83.7	100.0	100.0	100.0	100.0	99.9	99.9

Table 7: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric Clayton copula.

n	True	$\theta = 0.67$				$\theta = 2$				$\theta = 6$			
		$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aCl}_{\theta,0.8,1}$	7.0	5.8	4.3	4.8	8.8	7.8	2.2	5.2	17.8	28.5	16.3	31.1
	$\mathrm{aCl}_{\theta,0.6,1}$	6.7	3.8	3.6	4.1	8.4	10.3	5.6	6.9	36.9	50.3	38.7	49.5
	$\mathrm{aCl}_{\theta,0.4,1}$	5.2	4.7	4.0	4.3	11.1	10.6	7.7	8.2	36.4	40.6	33.9	33.8
	$\mathrm{aCl}_{\theta,0.2,1}$	6.0	5.5	4.5	4.8	6.7	8.0	4.6	4.8	14.9	15.9	12.3	12.5
100	$\mathrm{aCl}_{\theta,0.8,1}$	6.0	4.0	3.6	3.3	10.2	17.1	6.8	6.5	39.1	72.6	52.9	65.1
	$\mathrm{aCl}_{\theta,0.6,1}$	5.8	5.0	4.7	4.1	13.9	25.6	11.5	11.6	72.2	93.3	85.1	88.9
	$\mathrm{aCl}_{\theta,0.4,1}$	6.0	5.3	3.8	3.8	13.2	21.5	12.8	12.0	67.4	81.9	69.4	69.2
	$\mathrm{aCl}_{\theta,0.2,1}$	6.5	4.8	4.3	3.0	7.6	10.1	5.7	6.4	31.7	33.1	27.4	26.4
200	$\mathrm{aCl}_{\theta,0.8,1}$	4.6	5.3	3.8	3.7	16.3	39.2	13.5	14.3	71.6	98.6	93.2	95.4
	$\mathrm{aCl}_{\theta,0.6,1}$	7.0	8.1	4.8	4.9	27.7	57.2	28.4	28.4	96.5	100.0	99.8	99.9
	$\mathrm{aCl}_{\theta,0.4,1}$	6.3	7.1	4.7	4.5	23.6	46.0	24.5	25.4	93.3	99.1	97.3	97.2
	$\mathrm{aCl}_{\theta,0.2,1}$	5.8	6.0	4.6	4.7	12.1	16.9	11.4	10.4	54.8	61.8	51.2	50.8
400	$\mathrm{aCl}_{\theta,0.8,1}$	6.9	8.4	4.3	4.1	27.6	76.9	31.7	29.3	96.2	100.0	100.0	100.0
	$\mathrm{aCl}_{\theta,0.6,1}$	7.0	11.3	5.2	5.1	49.7	90.8	59.8	60.2	100.0	100.0	100.0	100.0
	$\mathrm{aCl}_{\theta,0.4,1}$	5.6	7.8	3.9	4.3	45.1	77.2	48.1	47.1	100.0	100.0	100.0	100.0
	$\mathrm{aCl}_{\theta,0.2,1}$	5.5	5.8	4.0	3.8	18.5	33.1	18.7	18.9	87.0	92.8	85.9	86.1

Table 8: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric normal copula.

n	True	$\theta = 0.38$				$\theta = 0.71$				$\theta = 0.92$			
		$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aN}_{\theta,0.8,1}$	4.9	3.9	3.3	3.9	6.5	4.6	3.8	5.6	25.4	20.6	12.0	21.2
	$\mathrm{aN}_{\theta,0.6,1}$	5.8	4.9	4.2	4.8	12.2	9.3	6.3	6.8	62.7	47.2	40.4	49.3
	$\mathrm{aN}_{\theta,0.4,1}$	6.5	5.4	4.4	5.1	18.2	13.1	10.4	11.5	69.9	51.4	47.1	48.2
	$\mathrm{aN}_{\theta,0.2,1}$	7.6	6.3	4.5	4.3	11.5	10.8	9.3	8.1	41.3	28.2	23.3	22.4
100	$\mathrm{aN}_{\theta,0.8,1}$	5.6	4.3	3.5	2.9	12.2	9.6	7.2	6.2	60.6	56.5	45.0	48.0
	$\mathrm{aN}_{\theta,0.6,1}$	6.6	5.4	3.8	3.8	25.6	23.1	15.4	13.3	95.4	92.7	89.8	89.4
	$\mathrm{aN}_{\theta,0.4,1}$	5.5	4.5	4.3	4.9	32.9	26.6	21.1	18.9	97.7	88.0	87.2	85.4
	$\mathrm{aN}_{\theta,0.2,1}$	6.2	5.6	3.8	3.6	24.1	19.3	12.9	12.7	80.0	52.4	51.0	49.3
200	$\mathrm{aN}_{\theta,0.8,1}$	4.4	5.4	4.6	4.4	19.1	18.6	12.1	10.0	93.6	94.8	91.7	88.4
	$\mathrm{aN}_{\theta,0.6,1}$	6.2	5.3	4.2	4.2	45.4	44.2	32.9	28.8	100.0	100.0	99.9	99.9
	$\mathrm{aN}_{\theta,0.4,1}$	7.4	6.1	4.5	4.3	57.0	53.3	39.2	37.0	100.0	99.9	99.8	99.9
	$\mathrm{aN}_{\theta,0.2,1}$	7.1	7.1	4.1	4.0	47.1	35.8	28.7	29.5	98.9	85.8	83.0	83.6
400	$\mathrm{aN}_{\theta,0.8,1}$	5.6	4.8	4.6	4.3	40.0	46.1	30.8	26.5	100.0	100.0	100.0	99.7
	$\mathrm{aN}_{\theta,0.6,1}$	7.5	6.5	6.3	5.3	80.5	83.2	67.3	62.3	100.0	100.0	100.0	100.0
	$\mathrm{aN}_{\theta,0.4,1}$	10.3	10.3	6.9	6.5	90.0	85.5	75.7	75.6	100.0	100.0	100.0	100.0
	$\mathrm{aN}_{\theta,0.2,1}$	9.9	9.3	7.1	6.9	76.4	63.2	55.2	52.6	100.0	99.5	99.6	99.6

Table 9: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric Frank copula.

n	True	$\theta = 2.37$				$\theta = 5.74$				$\theta = 14.14$			
		$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$	$\tilde{S}_{n,c}^{\mathrm{CFG}}$	$\tilde{S}_{n,c}^{\mathrm{P}}$	$T_n^{[1]}$	$T_n^{[2]}$
50	$\mathrm{aF}_{\theta,0.8,1}$	5.3	5.0	2.3	3.2	6.1	4.2	3.3	4.8	24.2	17.9	17.7	28.5
	$\mathrm{aF}_{\theta,0.6,1}$	6.3	4.9	4.9	5.6	16.1	14.8	10.7	11.7	53.0	51.4	49.5	56.2
	$\mathrm{aF}_{\theta,0.4,1}$	7.4	6.0	3.5	3.6	17.6	17.0	12.1	11.7	58.2	53.0	49.4	51.4
	$\mathrm{aF}_{\theta,0.2,1}$	6.2	5.4	4.8	4.4	9.7	10.7	6.9	6.5	29.9	25.8	21.2	19.2
100	$\mathrm{aF}_{\theta,0.8,1}$	5.7	5.0	3.9	3.6	10.9	9.4	8.4	7.8	49.8	47.9	57.9	64.1
	$\mathrm{aF}_{\theta,0.6,1}$	6.3	6.4	5.1	4.3	25.6	27.4	23.4	21.9	88.2	90.9	94.4	93.8
	$\mathrm{aF}_{\theta,0.4,1}$	7.3	6.9	5.7	5.4	29.6	32.3	26.4	25.8	89.6	89.7	90.1	88.6
	$\mathrm{aF}_{\theta,0.2,1}$	5.7	6.4	4.0	3.9	17.2	17.4	13.5	12.6	54.4	47.0	42.1	40.8
200	$\mathrm{aF}_{\theta,0.8,1}$	6.6	6.5	4.6	4.9	23.2	22.5	19.1	17.2	84.2	89.1	95.3	95.3
	$\mathrm{aF}_{\theta,0.6,1}$	9.4	9.9	6.9	6.3	46.7	55.8	49.8	49.3	99.9	100.0	100.0	100.0
	$\mathrm{aF}_{\theta,0.4,1}$	9.0	9.6	7.4	7.4	53.1	61.4	50.1	49.3	99.9	100.0	99.9	99.9
	$\mathrm{aF}_{\theta,0.2,1}$	7.7	7.8	6.0	5.6	30.8	32.0	25.8	25.8	85.2	82.3	78.4	78.0
400	$\mathrm{aF}_{\theta,0.8,1}$	7.8	8.5	7.0	7.0	40.1	46.1	45.4	39.8	99.0	99.7	100.0	100.0
	$\mathrm{aF}_{\theta,0.6,1}$	13.8	16.4	12.7	11.7	79.0	86.7	83.0	80.6	100.0	100.0	100.0	100.0
	$\mathrm{aF}_{\theta,0.4,1}$	15.4	18.1	13.9	13.9	83.6	91.1	85.5	84.8	100.0	100.0	100.0	100.0
	$\mathrm{aF}_{\theta,0.2,1}$	10.2	13.4	9.1	8.8	55.2	61.3	51.5	50.4	99.2	99.1	98.7	98.7

Table 10: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric version of the t copula with 4 degrees of freedom.

n	True	$\theta = 0.38$				$\theta = 0.71$				$\theta = 0.92$			
		$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$	$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$	$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$
50	at-$4_{\theta,0.8,1}$	7.4	5.3	4.9	4.4	8.6	6.4	5.1	6.3	30.8	22.4	14.5	27.7
	at-$4_{\theta,0.6,1}$	9.3	7.5	6.1	5.5	20.5	13.3	10.3	10.3	69.6	53.3	45.3	53.3
	at-$4_{\theta,0.4,1}$	9.0	7.0	4.6	5.6	23.6	16.1	10.8	12.1	73.8	49.1	43.5	45.3
	at-$4_{\theta,0.2,1}$	7.9	7.1	4.7	4.7	17.5	10.9	9.2	8.4	41.8	24.3	19.1	19.4
100	at-$4_{\theta,0.8,1}$	7.1	5.4	4.4	4.0	16.7	13.8	7.7	6.8	62.3	57.1	47.1	54.4
	at-$4_{\theta,0.6,1}$	10.2	8.5	6.5	6.0	36.0	28.9	24.2	22.5	96.2	90.7	88.6	89.4
	at-$4_{\theta,0.4,1}$	13.7	10.8	8.3	8.7	46.3	31.5	24.4	22.9	98.9	90.0	88.5	88.6
	at-$4_{\theta,0.2,1}$	11.0	8.2	6.6	7.3	35.3	21.5	17.1	16.4	79.6	51.4	48.2	46.6
200	at-$4_{\theta,0.8,1}$	8.5	7.0	5.6	5.2	28.5	30.3	17.5	16.3	93.3	94.8	86.8	89.3
	at-$4_{\theta,0.6,1}$	16.5	13.5	9.5	9.4	65.2	57.4	44.4	42.1	99.9	99.8	99.7	100.0
	at-$4_{\theta,0.4,1}$	22.2	17.5	14.0	12.3	79.2	63.8	53.4	52.2	100.0	99.8	99.9	100.0
	at-$4_{\theta,0.2,1}$	19.0	14.6	10.3	10.1	64.9	39.9	34.9	34.2	98.1	84.4	85.7	86.1
400	at-$4_{\theta,0.8,1}$	11.3	12.0	7.3	7.4	47.2	50.0	36.0	35.3	99.7	100.0	99.6	99.8
	at-$4_{\theta,0.6,1}$	26.2	26.6	16.0	16.0	90.9	89.0	79.1	78.0	100.0	100.0	100.0	100.0
	at-$4_{\theta,0.4,1}$	38.4	30.0	22.0	21.2	99.1	93.9	88.4	88.0	100.0	100.0	100.0	100.0
	at-$4_{\theta,0.2,1}$	34.7	22.5	18.8	18.2	92.6	69.9	67.6	67.5	100.0	99.6	99.5	99.6

Table 11: Percentage of rejection of H_0 computed from 1000 random samples of size 50, 100, 200 and 400 generated from the asymmetric Plackett copula.

n	True	$\theta = 3.14$				$\theta = 11.40$				$\theta = 68.45$			
		$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$	$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$	$\tilde{S}^{\mathrm{CFG}}_{n,c}$	$\tilde{S}^{\mathrm{P}}_{n,c}$	$T^{[1]}_n$	$T^{[2]}_n$
50	$\mathrm{aP}_{\theta,0.8,1}$	5.7	4.2	3.7	4.2	10.2	7.7	6.3	7.8	26.2	20.7	16.4	31.8
	$\mathrm{aP}_{\theta,0.6,1}$	6.3	5.0	4.2	4.7	19.0	14.0	12.0	13.5	61.7	50.0	46.7	57.2
	$\mathrm{aP}_{\theta,0.4,1}$	7.8	6.4	4.8	5.1	20.5	16.3	13.6	12.9	67.9	51.5	49.2	53.4
	$\mathrm{aP}_{\theta,0.2,1}$	6.2	6.1	3.5	4.0	12.3	8.7	7.9	7.6	34.9	21.7	19.2	19.4
100	$\mathrm{aP}_{\theta,0.8,1}$	6.7	5.9	5.5	5.7	17.4	13.0	10.2	11.0	56.8	56.9	52.6	63.2
	$\mathrm{aP}_{\theta,0.6,1}$	7.3	6.3	4.8	4.7	35.9	33.9	27.7	27.4	91.9	90.2	88.6	92.8
	$\mathrm{aP}_{\theta,0.4,1}$	8.3	9.1	6.2	6.0	38.4	36.7	30.6	29.0	95.5	88.5	87.3	89.1
	$\mathrm{aP}_{\theta,0.2,1}$	8.0	6.4	5.1	4.5	22.2	20.0	16.5	14.6	68.4	48.5	46.0	44.2
200	$\mathrm{aP}_{\theta,0.8,1}$	6.2	5.6	4.3	3.8	24.0	27.5	23.1	21.3	82.3	84.9	82.6	91.3
	$\mathrm{aP}_{\theta,0.6,1}$	10.7	10.5	7.3	6.4	61.6	64.8	57.4	57.0	99.8	99.5	99.7	99.9
	$\mathrm{aP}_{\theta,0.4,1}$	9.7	12.9	8.4	8.5	67.4	65.3	59.8	58.8	100.0	99.7	99.8	99.9
	$\mathrm{aP}_{\theta,0.2,1}$	7.9	9.4	6.8	7.2	42.2	37.6	33.5	32.3	95.0	83.4	80.8	81.4
400	$\mathrm{aP}_{\theta,0.8,1}$	8.9	9.4	8.2	7.8	44.5	49.2	45.1	44.5	98.2	99.3	99.1	99.7
	$\mathrm{aP}_{\theta,0.6,1}$	15.2	17.7	12.4	11.2	86.8	89.8	86.0	87.4	100.0	100.0	100.0	100.0
	$\mathrm{aP}_{\theta,0.4,1}$	18.9	21.7	17.6	16.8	94.4	93.5	92.3	92.1	100.0	100.0	100.0	100.0
	$\mathrm{aP}_{\theta,0.2,1}$	11.8	14.0	10.9	11.1	73.3	68.9	62.1	60.7	99.9	98.9	98.7	99.0