A Graphical Interpretation of the Choquet Integral (Short Communication)

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October 18, 1999

Abstract

We present a graphical interpretation of the Choquet integral, viewed as an aggregation operator, in the case of 2 elements. The interpretation relies on the interaction representation introduced by the author.

1 Introduction

The Choquet integral [1] has been introduced in the fuzzy measure community a decade ago by Murofushi and Sugeno [10], after its rediscovery by Weber in 1984 [13]. Since that time, the Choquet integral has gained an increasing popularity (see a survey of applications in [3, 5, 7]), where it is applied as a powerful aggregation operator over a finite set of elements.

However, as it is often the case with powerful or very general aggregation operators, a deep understanding of the aggregation mode of the Choquet integral is still lacking by most of the practitioners. This is the reason why simplified versions, or automatic learning procedures are used.

The author has introduced recently a new way of representing fuzzy measures, which shed light on the mechanism of the Choquet integral [6, 4]. This new representation is called the *interaction representation*.

In this short communication, we present a graphical interpretation of the Choquet integral when the universe is limited to two elements, based on the interaction representation. We believe that this interpretation will make clear the mechanism and the scope of the Choquet integral.

2 Background on the Choquet integral and the interaction representation

We consider a finite universal set $N = \{1, \ldots, n\}$, which can be thought as the index set of a set of criteria, attributes, experts, players, etc.

2.1 Fuzzy measures and interaction representation

Definition 1 *A* fuzzy measure *on N is a set function $\mu : \mathcal{P}(N) \longrightarrow [0,1]$ satisfying:*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$

(ii) $\mu(A) \leq \mu(B)$ *whenever* $A \subset B \subset N$ *(monotonicity).*

Usually, the meaning attributed to $\mu(A)$ is the *importance* or *power* of the *coalition* A (e.g. for making decision), where the word "coalition" refers to the domain of cooperative game theory. The monotonicity property means that adding a new element in a coalition cannot decrease the importance of the coalition.

Two particular cases of fuzzy measures are of interest here.

- A fuzzy measure μ is said to be *additive* if $\mu(A) = \sum_{i \in A} \mu(\{i\})$ for all $A \subset N$.
- A fuzzy measure is said to be *cardinal* if it depends only on the cardinality of sets, i.e. $\mu(A) = \mu(B)$ whenever $|A| = |B|$.

In cooperative game theory, an important concept is the concept of Shapley value of a game or *Shapley index* of a player [12]. For any element $i \in N$, the Shapley index of i is defined by:

$$\phi_i := \sum_{K \subset N \setminus i} \frac{(n - |K| - 1)!|K|!}{n!} [\mu(K \cup \{i\}) - \mu(K)], \qquad (1)$$

The Shapley value is then the vector $[\phi_1 \cdots \phi_n]$. Roughly speaking, the Shapley index ϕ_i computes the average contribution of element i in all coalitions, the average being weighted by a coefficient taking into acount the cardinality of the coalition. The Shapley value is uniquely characterized by a set of four axioms, among which we have $\sum_{i=1}^n \phi_i = \mu(N) = 1$, so that the sum of importance degrees is a constant. The idea to use the Shapley index for multicriteria decision making is due to Murofushi [8].

It is easy to show that for an additive measure, $\phi_i = \mu(\{i\})$ for all $i \in N$. Moreover, ϕ_i is constant for a cardinal measure.

A second important concept is the one of interaction index between two elements, proposed by Murofushi and Soneda [9]. It is defined as follows.

$$I_{ij} := \sum_{K \subset N \setminus \{i,j\}} \frac{(n-|K|-2)!|K|!}{(n-1)!}[\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)]. \tag{2}$$

It models the idea of interaction between elements in the following sense. Let us consider two elements i, j and their importance taken alone $\mu(\{i\}), \mu(\{j\})$, or put together $\mu(\{i,j\})$. Depending on the way the elements "cooperate", the quantity $\mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\})$ may be positive (which means that the cooperation is effective since working together is better than working separately), or negative (which means that the cooperation is harmful). The case of zero means that the elements act independently. I_{ij} computes the average of this key quantity, all possible coalitions being taken into account. Remark that for an additive measure, we have $I_{ij} = 0$ for all i, j.

Remarking the analogy between the Shapley index and the interaction index, Grabisch has proposed a general interaction index $I(A)$ defined for all coalitions (including the empty one), which is:

$$I(A) := \sum_{K \subset N \setminus A} \frac{(n-|K|-|A|)!|K|!}{(n-|A|+1)!} \sum_{B \subset A} (-1)^{|A|-|B|} \mu(K \cup B), \forall A \subset N. \tag{3}$$

Note that $I(\{i\}) = \phi_i$, and $I(\{i,j\}) = I_{ij}$. Also, it is easy to show that for an additive measure, $I(A) = 0$ whenever $|A| > 1$.

Formally, I can be considered as a linear transformation over the set of set functions, as the Möbius transform (see Denneberg and Grabisch for full details [2]), which is invertible. Specifically, the inverse of (3) reads:

$$\mu(A) = \sum_{B \subset N} \beta_{|A \cap B|}^{|B|} I(B), \quad \forall A \subset N \tag{4}$$

with

$$\beta_k^l := \sum_{j=0}^{k} \binom{k}{j} B_{l-j}, \tag{5}$$

where B_k are the Bernoulli numbers, with first terms $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, etc. First values of β_k^l are:

$k \setminus l$	0	1	2	3	4
0	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
1		$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{30}$
2			$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{15}$
3				0	$-\frac{1}{30}$
4					$-\frac{1}{30}$

Since the transformation is invertible, it is equivalent to define a fuzzy measure by giving $\mu(A)$ for all $A \subset N$, or $I(A)$ for all $A \subset N$, hence the term *interaction representation.*

Lastly, we introduce the concept of k-additive measure [6].

Definition 2 *A fuzzy measure is said to be a k-*additive measure *if $I(A) = 0$ for all coalitions of more than k elements, and there exists at least one A of exactly k elements such that $I(A) \neq 0$.*

A 1-additive measure is an ordinary (additive) measure in the classical sense. It can be shown that an equivalent definition is obtained if one replaces I by the Möbius transform (it is in fact the usual definition, see [6]).

2.2 The Choquet integral

Definition 3 *Let $f : X \longrightarrow \mathbb{R}^+$, and μ be a fuzzy measure. Let us denote $f(i)$ by f_i, $\forall i \in N$. The* Choquet integral *of f w.r.t. μ is defined by:*

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} (f_{(i)} - f_{(i-1)})\mu(\{(i), \ldots, (n)\})$$

where $(\cdot)$ means a permutation of the elements of N such that $f_{(1)} \leq \cdots \leq f_{(n)}$, and $f_{(0)} := 0$.

We identify from now on the set of functions $f : X \longrightarrow \mathbb{R}^+$ with $(\mathbb{R}^+)^n$. Two particular cases are of interest.

- If the fuzzy measure is additive, then the Choquet integral reduces to a weighted sum:

$$\mathcal{C}_\mu(a) = \sum_{i=1}^{n} \mu(\{i\})a_i, \forall a \in (\mathbb{R}^+)^n$$

- If the fuzzy measure is cardinal, then the Choquet integral reduces to an OWA operator [14]:

$$\mathcal{C}_\mu(a) = \sum_{i=1}^{n} w_i a_{(i)}, \forall a \in (\mathbb{R}^+)^n$$

with weights w_i defined by $w_i = \mu(A_{n-i+1}) - \mu(A_{n-i})$, $i = 1, \dots, n$, where A_i denotes any set of i elements (see [11]). In particular, the minimum and the maximum operators are recovered.

It is possible to express the Choquet integral in terms of the interaction representation [4]. The general expression is fairly complicated, but it reduces to an easily interpretable form in the case of (at most) 2-additive measures, which is, for any $a \in (\mathbb{R}^+)^n$:

$$\mathcal{C}_\mu(a) = \sum_{I_{ij}>0} (a_i \wedge a_j) I_{ij} + \sum_{I_{ij}<0} (a_i \vee a_j)|I_{ij}| + \sum_{i=1}^{n} a_i(\phi_i - \frac{1}{2}\sum_{j\neq i} |I_{ij}|), \tag{6}$$

with the property that $\phi_i - \frac{1}{2}\sum_{j\neq i}|I_{ij}| \geq 0$ for all i. It can be seen that the Choquet integral for 2-additive measures can be decomposed in a conjunctive, a disjunctive and an additive part, corresponding respectively to positive interaction indices, negative interaction indices, and the Shapley value. It makes clear the precise meaning of interaction when the Choquet integral is used.

3 Graphical interpretation when $n = 2$

We introduce now a graphical representation of the Choquet integral, in the case of 2 elements. Using (6) with $n = 2$, we get:

$$\mathrm{C}_\mu(a_1, a_2) = \begin{cases} (a_1 \wedge a_2) I_{12} + a_1(\phi_1 - \frac{1}{2}I_{12}) + a_2(\phi_2 - \frac{1}{2}I_{12}), & \text{if } I_{12} \geq 0 \\ (a_1 \vee a_2) I_{12} + a_1(\phi_1 + \frac{1}{2}I_{12}) + a_2(\phi_2 + \frac{1}{2}I_{12}), & \text{if } I_{12} \leq 0 \end{cases} \tag{7}$$

which is in fact the general expression for Choquet integral when $n = 2$, since in this case, at most 2-additive measures coincide with general fuzzy measures. Clearly, all possible Choquet integrals are obtained when ϕ_1, ϕ_2 and I_{12} vary on their domain. The following result defines the domain of the interaction index [6]:

Property 1 *A set of 2^n coefficients $I(A)$, $A \subset N$ corresponds to the interaction representation of a capacity if and only if* *(i)* $\sum_{A\subset N} B_{|A|} I(A) = 0$,

(ii) $\sum_{i\in N} I(\{i\}) = 1$,

(iii) $\sum_{A\subset N\setminus i} \beta^{|A|}_{|A\cap B|} I(A\cup\{i\}) \geq 0,\ \forall i\in N,\ \forall B\subset N\setminus\{i\}$

where the B_k's are the Bernoulli numbers, and $\beta^l_k := \sum_{j=0}^k \binom{k}{j} B_{l-j}, \quad k,l = 0,1,2,\ldots$.

Applying this to the case of 2 criteria, and recalling that $I(\{i\}) = \phi_i$, we get the following constraints:

$$\phi_1 - \frac{1}{2}I_{12} \geq 0$$
$$\phi_2 - \frac{1}{2}I_{12} \geq 0$$
$$\phi_1 + \frac{1}{2}I_{12} \geq 0$$
$$\phi_2 + \frac{1}{2}I_{12} \geq 0$$

Recalling that $\phi_1 + \phi_2 = 1$, this domain can be represented with the (ϕ_1, I_{12}) coordinates only, and is the shaded area on figure 1. In other words, the Choquet integral is the convex closure of the four vertices of Fig. 1.

In the remaining, we further study these vertices, and the two axes.

Let us consider the horizontal axis, where $I_{12} = 0$. In the case $n = 2$, this is equivalent to say that the measure is additive, and thus the Choquet integral is a weighted sum, with weights $\mu(\{1\})$ and $\mu(\{2\})$, which coincide with ϕ_1 and ϕ_2. This can be immediately seen from (7). In conclusion, the horizontal axis represent the set of all possible weighted sums, with left extremity corresponding to $\mathcal{C}_\mu(a_1, a_2) = a_2$ (i.e. element 2 is a dictator), and right extremity to the situation where element 1 is dictator.

Let us examine the vertical axis, where $\phi_1 = \phi_2 = \frac{1}{2}$. Assuming $I_{12} > 0$ (the converse case leads to the same result), formula (7) becomes:

$$\begin{aligned} \mathrm{C}_\mu(a_1, a_2) &= (a_1 \wedge a_2) I_{12} + \frac{1}{2}(a_1 + a_2)(1 - I_{12}) \\ &= a_{(1)} I_{12} + \frac{1}{2}(a_{(1)} + a_{(2)})(1 - I_{12}) \\ &= \frac{1}{2} a_{(1)}(1 + I_{12}) + \frac{1}{2} a_{(2)}(1 - I_{12}). \end{aligned} \tag{8}$$

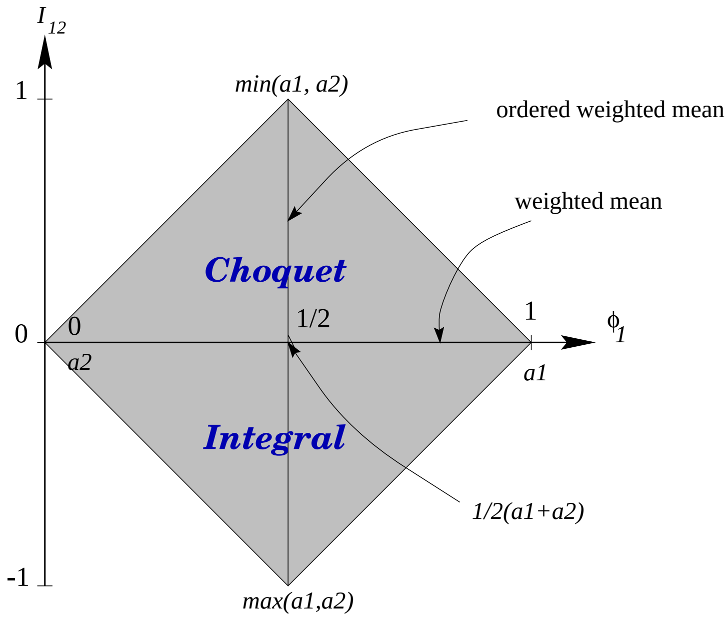

Figure 1: Interpretation in term of interaction index

We recognize here an OWA operator, with weights $\frac{1}{2}(1+I_{12})$ and $\frac{1}{2}(1-I_{12})$. Moreover, since any OWA operator is such that $\phi_1=\phi_2=\frac{1}{2}$ when $n=2$, the vertical axis is the locus of all possible OWA operators. The upper vertex ($I_{12}=1$) corresponds to the minimum operator, and the lower vertex to the maximum operator, as it can be seen form (8)

All these results are indicated on Fig. 1. It shows clearly that any Choquet integral can be written as a convex combination of a mininmum, a maximum, and two dictators, i.e. for any fuzzy measure μ, there exists positive numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha+\beta+\gamma+\delta=1$, and

$$\mathcal{C}_\mu(a_1,a_2)=\alpha(a_1\wedge a_2)+\beta(a_1\vee a_2)+\gamma a_1+\delta a_2.$$

The reciprocal holds also: any convex combination of a minimum, a maximum and 2 dictators is a Choquet integral.

4 Conclusion

This short paper has given a clear understanding of the Choquet integral viewed as an aggregation oeprator, using a graphical representation. Our result is of course only valid when $n=2$. The case of higher dimensionality seems to be much more complex. For $n=3$, we have 6 free coefficients, namely $\phi_1, \phi_2, I_{12}, I_{13}, I_{23}$ and $I(\{1,2,3\})$, and this is no more graphically representable in the general case.

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